# A MATHEMATICAL THEORY OF QUANTUM SHEAF COHOMOLOGY

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ABSTRACT. The purpose of this paper is to present a mathematical theory of the half-twisted (0,2) gauged linear sigma model and its correlation functions that agrees with and extends results from physics. The theory is associated to a smooth projective toric variety X and a deformation  $\mathcal{E}$  of its tangent bundle  $T_X$ . It gives a quantum deformation of the cohomology ring of the exterior algebra of  $\mathcal{E}^*$ . We prove that in the general case, the correlation functions are independent of 'nonlinear' deformations. We derive quantum sheaf cohomology relations that correctly specialize to the ordinary quantum cohomology relations described by Batyrev in the special case  $\mathcal{E} = T_X$ .

#### 1. Introduction

The gauged linear sigma model (GLSM) was introduced in [Wit93] as a quantum field theory that is closely related to the nonlinear sigma model (NLSM), but easier to analyze for both (2,2) and (0,2) versions. Quantum cohomology relations for the (2,2) GLSM were described in [Bat93] and elaborated on by [MP95]. In this paper, we present a mathematical theory for the (0,2) GLSM and derive analogous results. To put this work in context, we give some background and motivation from physics before focusing on the mathematical formulation. More details from the physics perspective are given in the paper [DGKS11] by the authors.

The (0,2) NLSM is a physical theory associated to a Calabi-Yau threefold X and a vector bundle  $\mathcal{E}$  on X satisfying  $c_1(\mathcal{E})=0$  and the Green-Schwarz anomaly cancellation condition  $c_2(\mathcal{E})=c_2(T_X)$ . The half-twisted (0,2) NLSM (sometimes called the A/2 model) is a closely-related but simpler theory that can be constructed in a much more general situation.

Consider a compact Kähler manifold X with a holomorphic vector bundle  $\mathcal{E} \to X$  satisfying the equalities

$$(i) c_1(\mathcal{E}) = c_1(T_X),$$

(ii) 
$$c_2(\mathcal{E}) = c_2(T_X)$$

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in the cohomology of X. As these conditions extend the usual Green-Schwarz anomaly cancellation condition of heterotic string theory we will call such a bundle omalous (i.e. not anomalous). Given an omalous bundle  $\mathcal{E}$  on X, the half-twisted (0,2) NLSM can be defined as a physical theory associated to maps from a genuszero Riemann surface  $\Sigma$  to X, with fermions associated to  $\mathcal{E}$ . This quantum field theory possesses a "quasi-topological subsector"; a subalgebra of vertex operators conjectured to be independent of the complex structure on X and referred to as the quantum sheaf cohomology of  $\mathcal{E} \to X$  [ABS04, ADE06]. The operators in the quasi-topological sector are in one-to-one correspondence with the sheaf cohomology  $\bigoplus_{p,q} H^q(X, \bigwedge^p \mathcal{E}^\vee)$ .

Ignoring quantum corrections, the product of operators corresponds to the cup product of corresponding classes

$$H^{q}(X, \bigwedge^{p} \mathcal{E}^{\vee}) \otimes H^{q'}(X, \bigwedge^{p'} \mathcal{E}^{\vee}) \xrightarrow{\cup} H^{q+q'}(X, \bigwedge^{p+p'} \mathcal{E}^{\vee}),$$

where the cup product  $\cup$  refers to the usual cup product on cohomology followed by the product in the exterior algebra of  $\mathcal{E}^{\vee}$ . In the full quantum theory, instanton corrections modify the product, but for now we discuss the classical algebra, which we call the *polymology* of  $\mathcal{E}$ .

**Definition 1.1.** The polymology of a vector bundle  $\mathcal{E}$  is the associative algebra

(1) 
$$H_{\mathcal{E}}^*(X) := \bigoplus_{p,q} H^q(X, \bigwedge^p \mathcal{E}^{\vee})$$

equipped with the cup product.

If  $\mathcal{E} = T_X$ , i.e. if the (0,2) theory is actually a (2,2) theory, then the polymology is canonically isomorphic to the ordinary cohomology of X by Hodge theory.

The polymology can be defined for any vector bundle, but if  $\mathcal{E}$  is omalous, a choice of isomorphism  $\det \mathcal{E}^{\vee} \simeq \omega_X$  induces an isomorphism  $\psi : H^n(X, \det \mathcal{E}^{\vee}) \simeq H^n(X, \omega_X)$ , where n is the dimension of X. This isomorphism in turn induces a pairing

(2) 
$$(\alpha, \beta) = \int_X \psi(\alpha \cup \beta)$$

satisfying  $(\alpha \cup \beta, \gamma) = (\alpha, \beta \cup \gamma)$ , which is perfect by Serre duality:

(3) 
$$H^{p}(X, \bigwedge^{q} \mathcal{E}^{\vee})^{\vee} \simeq H^{n-p}(X, \bigwedge^{q} \mathcal{E} \otimes \omega_{X})$$
$$\simeq H^{n-p}(X, \bigwedge^{q} \mathcal{E} \otimes \bigwedge^{n} \mathcal{E}^{\vee})$$
$$\simeq H^{n-p}(X, \bigwedge^{n-q} \mathcal{E}^{\vee}).$$

It follows easily that the polymology of an omalous vector bundle admits the structure of a bigraded Frobenius algebra.

The classical correlation functions can be identified with the pairing (2). Note that as  $H^n(X, \bigwedge^n \mathcal{E}^{\vee})$  is isomorphic to the complex numbers, any such isomorphism defines the trace in the Frobenius algebra structure. However, this isomorphism is

not canonical. We deal with this normalization issue by simply defining the classical correlation functions to live in the one-dimensional vector space  $H^n(X, \bigwedge^n \mathcal{E}^{\vee})$ .

Before discussing the quantum sheaf cohomology of  $(X, \mathcal{E})$ , a brief discussion is in order about two relevant quantum field theories: the (2,2) NLSM and the (2,2) GLSM. A little later we will discuss the (0,2) GLSM and the (0,2) NLSM.

Given a smooth projective variety X, the quantum corrections to any of these quantum field theories can be computed perturbatively using a compactification of the space of holomorphic maps  $f: \mathbb{P}^1 \to X$  with  $f_*[\mathbb{P}^1] = \beta$ , for each  $\beta \in H_2(X,\mathbb{Z})$ , and performing an integration over this compactification. The (2,2) NLSM is well-understood mathematically as ordinary quantum cohomology, with the appropriate compactification being the moduli space  $\overline{M}_{0,3}(X,\beta)$  of genus 0 stable maps of class  $\beta$ . If X is a toric variety, there is the linear sigma model moduli space  $X_\beta$ , which is a toric compactification used for the (2,2) GLSM [Wit93, MP95], leading to a quantum cohomology ring whose structure was described by Batyrev in [Bat93]. Either of these quantum cohomology rings are deformations of the ordinary cohomology ring  $H^*(X)$ . A comparison between Batyrev's quantum cohomology ring and the ordinary quantum cohomology ring follows from [Giv98] and is described in [CK00]. These two cohomology rings become identified after a change in variables (the mirror map). The Batyrev quantum cohomology ring is identical to the usual quantum cohomology ring if X is Fano.

The quantum sheaf cohomology described in this paper arises from the half-twisted (0,2) GLSM, and extends Batyrev's quantum cohomology ring. We will also explain how the identical moduli space  $X_{\beta}$  used in the (2,2) GLSM moduli space can also be used to describe the (0,2) GLSM, independent of  $\mathcal{E}$ .

Quantum sheaf cohomology is a quantum deformation of the Frobenius algebra structure on the polymology of  $(X, \mathcal{E})$ . This is analogous to either of the two versions of the quantum cohomology of X mentioned above.

Physics tells us from the half-twisted (0,2) NLSM that a quantum sheaf cohomology ring is associated to  $(X, \mathcal{E})$  for any omalous vector bundle  $\mathcal{E}$  on X. Unfortunately, a mathematically-precise version of such a theory does not yet exist. However, physical arguments providing an approach to such a mathematical version are given in [KS06]. Furthermore, one can speculate that the relevant compactification of the space of maps will be  $\overline{M}_{0,3}(X,\beta)$  as in the (2,2) NLSM, independent of the choice of bundle  $\mathcal{E}$ . One can also speculate that the GLSM and NLSM versions of quantum sheaf cohomology will be identified by a change of variables analogous to the mirror map.

We now describe the ingredients of the half-twisted (0,2) GLSM. Although such theories are more general, we will only describe them in the situation we consider. Let X be a smooth projective toric variety, and let  $\mathcal{E}$  be a deformation of the tangent bundle  $T_X$  arising from a deformation of the toric Euler sequence (to be described in (8) below). The half-twisted (0,2) GLSM is associated to  $(X,\mathcal{E})$ , and has a

quasi-topological sector whose operators are generated by the symmetric algebra on  $H^2(X,\mathbb{C})$  and become isomorphic to  $H^*_{\mathcal{E}}(X)$  in the classical limit. In physics, the quasi-topological sector arises as the set of operators lying in the kernel of the scalar supercharge whose holomorphic conformal weight vanishes.

One of our main results is the calculation of the classical polymology of  $(X, \mathcal{E})$  in Theorem 3.16. The algebra  $H_{\mathcal{E}}^*(X)$  is naturally a quotient of this symmetric algebra, relating the GLSM to the NLSM. To each primitive collection in the toric variety X is associated a generator of the Stanley-Reisner ideal  $SR(X, \mathcal{E})$ . The classical polymology is the quotient of this symmetric algebra by  $SR(X, \mathcal{E})$ .

Since  $X_{\beta}$  is itself toric, the same result can be used to compute quantum corrections in all instanton sectors. We introduce a direct system of polymologies in Section 4.3 which allows us to sum over the instanton sectors and rigorously define the quantum sheaf cohomology ring by abstracting the physical notion of an operator. For each primitive collection, a quantum deformation of the corresponding Stanley-Reisner generators can be written down (58), as proposed in the physics literature. Our final main result is:

**Theorem 4.10.** The quantum cohomology relations (58) hold for all primitive collections K.

Here is an outline of the rest of the paper.

We begin Section 2 by recalling standard concepts and notation from toric geometry. Then we recall the toric Euler sequence of a smooth toric variety X, which gives a presentation of its tangent bundle. Deformations of this sequence are presentations of deformations  $\mathcal{E}$  of the tangent bundle, which complete the input data needed to define the (0,2) GLSM. We then introduce a generalized Koszul complex that plays a fundamental role in our analysis. We conclude the section with the computation of the sheaf cohomology of certain T-invariant divisors, which enable us to chase through exact cohomology sequences associated with the generalized Koszul complex.

In Section 3 we compute the polymology of  $(X, \mathcal{E})$ . Let  $W = H^2(X, \mathbb{C})$  and let  $K \subset \Sigma(1)$  be a primitive collection, where  $\Sigma(1)$  as usual denotes the edges of the fan for X. By a diagram chase, we associate to each K an element  $Q_K \in \operatorname{Sym}^k W$ , where k = |K|. We then define the Stanley-Reisner ideal  $\operatorname{SR}(X, \mathcal{E})$  of  $\mathcal{E}$  to be the ideal in  $\operatorname{Sym}^* W$  generated by the  $Q_K$ . The main result of Section 3 is that the polymology of  $\mathcal{E}$  is isomorphic to the quotient of  $\operatorname{Sym}^* W$  by the Stanley-Reisner ideal of  $\mathcal{E}$  (Theorem 3.16).

In Section 4 we describe the GLSM moduli space  $X_{\beta}$  associated to an effective class  $\beta \in H_2(X, \mathbb{Z})$  and an induced vector bundle  $\mathcal{E}_{\beta}$  on  $X_{\beta}$ . The correlation functions are defined in terms of the polymology of  $(X_{\beta}, \mathcal{E}_{\beta})$ . We show that the polymology of  $(X_{\beta}, \mathcal{E}_{\beta})$  is a quotient of Sym<sup>\*</sup>W by an ideal generated by powers of the  $Q_K$  given by (49). We then introduce a direct system of polymologies that will allow

us to compare correlation functions for different  $\beta$ , and then define the correlation functions after introducing four-fermi terms in (54) that play a role for the (0,2) GLSM similar to that of the virtual fundamental class of Gromov-Witten theory. Finally, we define the quantum sheaf cohomology ring abstractly and then compute the quantum cohomology relations in Theorem 4.10. In [MM09], predictions were made for the image of the relations in a localization of the ring (following a standard procedure in the physics literature). It is straightforward to verify that our relations descend to the relations of [MM09] in that localization. See [DGKS11] for details.

We now turn to toric geometry to elaborate on the set-up.

## 2. The toric setting

We start by fixing some notation associated to toric varieties. A good general reference for toric varieties is [CLS11].

Let  $X = X_{\Sigma}$  be a smooth projective toric variety of dimension n with fan  $\Sigma$  whose support lies in  $N_{\mathbb{R}} \simeq \mathbb{R}^n$ , where N is the lattice of one-parameter subgroups of the dense torus of X. We will denote by M the lattice of characters dual to N, by  $\Sigma(1)$  the set of one-dimensional cones of the fan, and we will write  $\bigoplus_{\rho}$  and  $\sum_{\rho}$  in place of  $\bigoplus_{\rho \in \Sigma(1)}$  and  $\sum_{\rho \in \Sigma(1)}$ , respectively. To each  $\rho \in \Sigma(1)$  is associated a torus-invariant Weil divisor denoted  $D_{\rho}$ , a unique minimal generator  $v_{\rho}$  of the semigroup  $N \cap \rho$ , and a canonical section  $x_{\rho} \in H^{0}(X, \mathcal{O}_{X}(D_{\rho}))$ . These canonical sections freely generate the homogeneous coordinate ring of X:

(4) 
$$S := \mathbb{C}[x_{\rho} \mid \rho \in \Sigma(1)].$$

The homogenous coordinate ring S has a natural grading by Pic(X), which assigns to  $x_{\rho}$  the degree  $[D_{\rho}] \in \text{Pic}(X)$ .

For each T-invariant Weil divisor  $D = \sum_{\rho} a_{\rho} D_{\rho}$  we have a natural isomorphism

(5) 
$$S_{[D]} \simeq H^0(X, \mathcal{O}_X(D)),$$

where as usual,  $S_{[D]}$  denotes the graded piece of S of degree [D]. To describe this isomorphism, we associate to D the polytope

$$\Delta_D = \{ m \in M_{\mathbb{R}} \mid \langle m, v_{\rho} \rangle \ge -a_{\rho} \text{ for all } \rho \in \Sigma(1) \}.$$

Then the isomorphism (5) is conveniently described by the identification of basis elements

(6) 
$$\prod_{\rho} x_{\rho}^{\langle m, v_{\rho} \rangle + a_{\rho}} \leftrightarrow \chi^{m},$$

where m ranges over  $\Delta_D \cap M$  and  $\chi^m$  is the character associated to  $m \in M$ , thought of as a meromorphic function with at worst poles on D. Note in particular that for each  $\rho$  the trivial character  $\chi^0$  (i.e. the constant function 1) is the section of  $\mathcal{O}_X(D_\rho)$  associated to  $x_\rho$  via the isomorphism (5) for  $D = D_\rho$ .

Since X is smooth and toric, the class group of Weil divisors, the Picard group, and the integral cohomology are all isomorphic. We associate to each  $m \in M$  the element  $e_m \in \mathbb{Z}^{\Sigma(1)}$  defined by  $e_m(\rho) = \langle m, v_\rho \rangle$ . Then  $\operatorname{Pic}(X)$  admits a presentation as

(7) 
$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \operatorname{Pic}(X) \to 0,$$

where the first non-trivial morphism is  $m \mapsto e_m$  and the basis element of  $\mathbb{Z}^{\Sigma(1)}$  dual to  $\rho$  maps to  $[D_{\rho}]$ .

Let  $W = H^2(X, \mathbb{C})$ . Then the tangent bundle of X fits into a short exact sequence known as the toric Euler sequence:

$$(8) 0 \longrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} W^{\vee} \xrightarrow{E_0} \bigoplus_{\rho} \mathcal{O}_X(D_{\rho}) \longrightarrow T_X \longrightarrow 0.$$

Thinking of  $E_0$  as an element of  $\bigoplus_{\rho} S_{[D_{\rho}]} \otimes W$ , the  $\rho^{\text{th}}$  component of  $E_0$  is  $x_{\rho} \otimes [D_{\rho}]$ .

Recall that a collection of edges  $K \subset \Sigma(1)$  is a primitive collection if K does not span any cone in  $\Sigma$ , but every proper subcollection of K does. Equivalently, the intersection of the divisors  $D_{\rho}$  with  $\rho \in K$  is empty, but the intersection of any proper subset of these divisors is nonempty. Following the presentation of [CK00, §8.1.2], we define two ideals in the homogeneous coordinate ring S:

(9) 
$$P(X) = \left(\sum_{\rho} \langle m, v_{\rho} \rangle x_{\rho} \mid m \in M\right)$$
$$SR(X) = \left(\prod_{\rho \in K} x_{\rho} \mid K \text{ a primitive collection of } \Sigma\right).$$

The former is the ideal of linear equivalences, so that

(10) 
$$\operatorname{Sym}(W) \simeq S/P(X).$$

The second ideal in (9) is known as the *Stanley-Reisner* ideal of X. It is well known [Ful93, Oda88] that there is an isomorphism of  $\mathbb{Z}$ -graded algebras

(11) 
$$S/(P(X) + SR(X)) \simeq H^*(X, \mathbb{C})$$

induced by sending a generator  $x_{\rho}$  of S to  $[D_{\rho}]$ .

For later use, we recall the description of toric varieties as quotients. There is a natural action of  $G = \operatorname{Hom}(\operatorname{Pic}(X), \mathbb{C}^*)$  on  $\mathbb{C}^{\Sigma(1)}$  following from the inclusion  $G \subset \operatorname{Hom}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*)$  derived from (7). For each  $\sigma \in \Sigma$  define

$$(12) x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_{\rho}$$

and the irrelevant ideal

(13) 
$$B(\Sigma) = (x^{\hat{\sigma}} \mid \sigma \in \Sigma).$$

Thinking of the  $x_{\rho}$  as coordinate functions on  $\mathbb{C}^{\Sigma(1)}$ , we define the subset  $Z(\Sigma) \subset \mathbb{C}^{\Sigma(1)}$  as the vanishing locus of the irrelevant ideal. Then  $Z(\Sigma)$  is G-invariant and

(14) 
$$X = \left(\mathbb{C}^{\Sigma(1)} - Z(\Sigma)\right)/G.$$

For later use, we note that it is well known that  $Z(\Sigma)$  can be described in terms of primitive collections. If  $K = \{\rho_1, \dots, \rho_k\}$  is a primitive collection, let  $L_K \subset \mathbb{C}^{\Sigma(1)}$  be the linear subspace defined by  $x_{\rho_1} = \dots = x_{\rho_k} = 0$ . Then

$$(15) Z(\Sigma) = \cup_K L_K,$$

where the union is taken over all primitive collections K. The fan  $\Sigma$  can also be recovered from the set of primitive collections as the set of cones spanned by collections of edges that do not contain any primitive collection. See [CLS11] for example.

It will also be useful to note that Pic(X) can be recovered from G as

(16) 
$$\operatorname{Pic}(X) \simeq \operatorname{Hom}(G, \mathbb{C}^*),$$

by duality for finitely-generated abelian groups.

2.1. Toric deformations of the tangent bundle. To define a half-twisted (0,2) GLSM, we need a presentation of a vector bundle  $\mathcal{E}$  obtained from (8) by simply changing the map  $E_0$ :

$$(17) 0 \longrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} W^{\vee} \stackrel{E}{\longrightarrow} \bigoplus_{\rho} \mathcal{O}_X(D_{\rho}) \longrightarrow \mathcal{E} \longrightarrow 0.$$

Both the bundle and the presentation are required to define the GLSM. We will sometimes abuse terminology slightly by referring to the bundle  $\mathcal{E}$  as a toric deformation of the tangent bundle, but we always have a fixed presentation (17) in mind. Specifying a map E is not sufficient; it is required that the cokernel  $\mathcal{E}$  of E is locally free, or equivalently that

(18) 
$$E^{t}: \oplus_{\rho} \mathcal{O}(-D_{\rho}) \to W \otimes \mathcal{O}_{X}$$

is surjective.

As with  $E_0$ , the map E can be viewed as a section of  $\bigoplus_{\rho} H^0(X, \mathcal{O}(D_{\rho})) \otimes W = \bigoplus_{\rho} S_{[D_{\rho}]} \otimes W$ .

The components  $E_{\rho}$  of E can be thought of as W-valued sections of  $\mathcal{O}(D_{\rho})$ . We will sometimes express these sections as

$$E_{\rho} = \sum_{m \in \Delta_{D_{\rho}} \cap M} a_{\rho m} \chi^{m},$$

where  $a_{\rho m} \in W$ , or as

$$E_{\rho} = \sum_{m \in \Delta_{D_{\rho}} \cap M} a_{\rho m} x_{\rho} \prod_{\rho'} x_{\rho'}^{\langle m, v_{\rho}' \rangle}$$

using the identification (6).

The terms  $a_{\rho m} \chi^m$  with  $x_{\rho} \prod_{\rho'} x_{\rho'}^{\langle m, v_{\rho'}' \rangle}$  a linear monomial in the homogeneous coordinates will be called a *linear term*; the other terms will be called the *nonlinear terms*. A toric deformation of the tangent bundle containing only linear terms will be called a *linear deformation*. Linear deformations play a significant role in

physicists' analyses of quantum sheaf cohomology [KS06, GK10, MM08, MM09] in localized rings, and we will see that they capture the essence of the polymology associated to a toric deformation of the tangent bundle.

Throughout, we will make extensive use of a generalized Koszul complex associated to deformations of the toric Euler sequence. In order to simplify notation, put  $Z = \bigoplus_{\rho} \mathcal{O}(-D_{\rho})$  and then for  $0 \le k$  and  $0 \le j \le k$  define

$$\mathfrak{Z}_{j}^{(k)} := \bigwedge^{j} Z \otimes \operatorname{Sym}^{k-j} W.$$

Let  $\mathcal{E}$  be a deformation of the tangent bundle in the preceding sense. The dual of the exact sequence (17) induces an injection  $\bigwedge^k \mathcal{E}^{\vee} \to \bigwedge^k Z$  and maps  $\alpha_j : \mathfrak{Z}_j^{(k)} \to \mathfrak{Z}_{j-1}^{(k)}$  defined as

$$(19) \ \alpha_j : (z_1 \wedge \cdots \wedge z_j) \otimes s \mapsto \sum_{\ell=1}^j (-1)^{\ell-1} (z_1 \wedge \cdots \wedge \hat{z}_\ell \wedge \cdots \wedge z_j) \otimes [E^{\vee}(z_\ell) \odot s],$$

where E is the injection in (17) and  $\odot$  is multiplication in Sym<sup>\*</sup>W. These maps may be arranged into an exact sequence

$$(20) 0 \longrightarrow \bigwedge^{k} \mathcal{E}^{\vee} \longrightarrow \mathfrak{Z}_{k}^{(k)} \longrightarrow \mathfrak{Z}_{k-1}^{(k)} \longrightarrow \cdots \longrightarrow \mathfrak{Z}_{1}^{(k)} \longrightarrow \mathfrak{Z}_{0}^{(k)} \longrightarrow 0.$$

Exactness follows since the maps in (20) are natural and the analogous complex formed from a short exact sequence of vector spaces is easily seen to be exact.

2.2. A vanishing result for certain toric line bundles. We will make extensive use of the line bundles  $\mathcal{O}_X(D)$  associated to T-invariant divisors D on X, and in particular Theorem 2.1 below, which appears in [Dem70] and is reproduced here for convenience.

Consider a Weil divisor  $D = \sum_{\rho} a_{\rho} D_{\rho}$  and define

$$\Sigma_{D,m} = \{ \rho \in \Sigma(1) \mid \langle m, v_{\rho} \rangle < -a_{\rho} \}.$$

Then let  $V_{D,m}^{-1}$  be the union of polytopes in  $N_{\mathbb{R}}$ 

$$V_{D,m} = \bigcup_{\substack{\sigma \in \Sigma \\ \sigma(1) \subset \Sigma_{D,m}}} \operatorname{Conv}(v_{\rho} \mid \rho \in \sigma).$$

Since  $\mathcal{O}_X(-D)$  is a torus-equivariant bundle,  $H^j(X,\mathcal{O}_X(-D))$  decomposes as a direct sum of weight spaces  $H^j(X,\mathcal{O}_X(-D))_m$  with  $m \in M$ .

**Theorem 2.1** (Proposition 6 of [Dem70]). Let  $D = \sum_{\rho} a_{\rho} D_{\rho}$  be a T-invariant Weil divisor on X. For  $m \in M$  and  $p \geq 0$ ,

$$H^p(X, \mathcal{O}_X(D))_m \simeq \widetilde{H}^{p-1}(V_{D,m}, \mathbb{C}).$$

Here  $\widetilde{H}$  denotes the reduced cohomology. Consider a subset  $K \subset \Sigma(1)$  and set  $D_K = \sum_{\rho \in K} D_{\rho}$ .

<sup>&</sup>lt;sup>1</sup>We are following the notation of [CLS11], adapting it slightly; our  $V_{D,m}$  matches their  $V_{D,m}^{\text{simp}}$ .

**Lemma 2.2.** For all j and all  $K \subset \Sigma(1)$ ,  $H^j(X, \mathcal{O}_X(-D_K)) = H^j(X, \mathcal{O}_X(-D_K))_0$ . That is, the cohomology of  $\mathcal{O}_X(-D_K)$  is purely of weight 0.

*Proof.* By Theorem 2.1,  $H^j(X, \mathcal{O}_X(-D_K))_m$  is the reduced cohomology of the topological space  $V_{-D_K,m}$  obtained as follows:  $\Sigma$  determines a simplicial complex whose faces are  $\operatorname{Conv}(v_\rho|\rho\in\sigma(1))$  for  $\sigma\in\Sigma$ .  $V_{-D_K,m}$  is the subcomplex corresponding to those  $\sigma$  such that  $\sigma(1)$  is contained in:

$$\Sigma_{-D_K,m} := \{\rho \in K | \langle m, v_\rho \rangle \leq 0\} \cup \{\rho \notin K | \langle m, v_\rho \rangle < 0\}.$$

Here we use the fact that the coefficients of the  $D_{\rho}$  in  $-D_K$  are either 0 or -1. The set  $\Sigma_{-D_K,m}$  is invariant under rescaling m by a positive integer, and therefore so are  $V_{D,m}$  and  $H^j(X, \mathcal{O}_X(-D_K))_m$ . If the latter were non-vanishing for some non-zero m,  $H^j(X, \mathcal{O}_X(-D_K))$  would not be finite dimensional, contradicting the projectivity of X.

**Proposition 2.3.** Let  $\Sigma$  be a simplicial fan and  $K \subset \Sigma(1)$ . Setting k = |K| and  $D_K = \sum_{a \in K} D_a$  as before, we have that

- i) For all  $\ell \geq k$ ,  $H^{\ell}(X, \mathcal{O}_X(-D_K)) = 0$
- ii) If  $\bigcap_{\rho \in K} D_{\rho} \neq \emptyset$ , then for all  $\ell \in \mathbb{Z}$ ,  $H^{\ell}(X, \mathcal{O}_X(-D_K)) = 0$ .
- iii) If K is a primitive collection,

$$H^{\ell}(X, \mathcal{O}_X(-D_K)) \simeq \begin{cases} \mathbb{C} & \ell = k-1 \\ 0 & otherwise \end{cases}$$

iv) If K is not a primitive collection, then  $H^{k-1}(X, \mathcal{O}_X(-D_K)) = 0$ .

*Proof.* We use Theorem 2.1 and the notation therein throughout. By Lemma 2.2, we need only consider the torus-invariant part of the cohomology. The relevant set of one-cones is  $\Sigma_{-D_K,0} = K$ .

- i)  $V_{-D_K,0}$  is contained in the convex hull of k points and so can never contain a non-contractible k-1 cycle. Similarly, it does not contain any  $\ell$  cycles with  $\ell > k-1$ .
- ii) If the intersection is nonempty, cone $\{v_{\rho} \mid \rho \in K\} \in \Sigma$  and the  $v_{\rho}$  are linearly independent since  $\Sigma$  is simplicial: thus  $V_{-D_{K},0}$  is a k-1 simplex.
- iii) Consider a primitive collection K. Since K is primitive, every proper subset of K spans a cone in  $\Sigma$ , so the simplicial complex takes the form

$$V_{-D_K,0} = \bigcup_{\rho' \in K} \operatorname{Conv}(v_\rho \mid \rho \in K, \rho \neq \rho').$$

This set is precisely the boundary of the (k-1)-simplex  $\operatorname{Conv}(v_{\rho} \mid \rho \in K)$ , so that  $V_{-D_K,0}$  is homeomorphic to the (k-2) sphere and the last claim follows immediately.

iv) If K is not a primitive collection, we need only consider the situation where  $\bigcap_{\rho \in K} D_{\rho} = \emptyset$ . Then by comparison to the analysis in iii) above, we see that either  $V_{-D_K,0}$  has dimension strictly less than k-2, or it has dimension k-2 and

is homeomorphic to a proper subcomplex of the above simplicial triangulation of  $S^{k-2}$ . Either way we conclude that  $\widetilde{H}^{k-2}(V_{-D_K,0}) = 0$  and we are done.

Remark 2.4. An immediate consequence of ii) is that for all  $\ell \in \mathbb{Z}$  and  $\rho \in \Sigma(1)$ ,  $H^{\ell}(X, \mathcal{O}_X(-D_{\rho})) = 0$ .

3.  $H^*_{\mathcal{E}}(X)$  for toric deformations of the tangent bundle

In this section, we study the algebra  $H_{\mathcal{E}}^*(X)$ , showing that as a bigraded vector space it is isomorphic to  $H^*(X,\mathbb{C})$ . Multiplicatively it is generated under the cup product by elements of  $H^1(X,\mathcal{E}^\vee)$ . We show that the relations among the generators may be given explicitly by defining an ideal analogous to SR(X). Some of the results in this section are not used elsewhere in this paper, but we include them since they could be useful in applications to the NLSM.

3.1. Graded components. We begin our study of  $H_{\mathcal{E}}^*(X)$  by elucidating its vector space structure. In particular, we show that it is diagonal, in the sense that its graded components  $H^q(X, \bigwedge^p \mathcal{E}^{\vee})$  vanish unless p = q.

**Proposition 3.1.** Let  $\mathcal{E}$  be a locally-free toric Euler sequence deformation. Then for any p and  $q \neq p$ ,

$$H^q(X, \bigwedge^p \mathcal{E}^{\vee}) = 0.$$

*Proof.* For q = 0, the proposition holds trivially, since  $\bigwedge^p \mathcal{E}^{\vee} \subset \bigwedge^p Z$  and  $H^0(X, \bigwedge^p Z) = 0$ . Thus, assume that q > 0.

Consider the exact sequence in (20). For each  $1 \le j \le p-1$  define

(21) 
$$S_j^{(p)} = \ker\left(\mathfrak{Z}_j^{(p)} \to \mathfrak{Z}_{j-1}^{(p)}\right),$$

and set  $S_0^{(p)} := \operatorname{Sym}^p W \otimes \mathcal{O}_X$  and  $S_p^{(p)} := \bigwedge^p \mathcal{E}^{\vee}$ . Then, for each  $0 < \ell \leq p$ , we have a short exact sequence

$$(22) 0 \longrightarrow S_{\ell}^{(p)} \longrightarrow \mathcal{J}_{\ell}^{(p)} \longrightarrow S_{\ell-1}^{(p)} \longrightarrow 0.$$

We first show that for q > p,  $H^q(X, \bigwedge^p \mathcal{E}^{\vee}) = 0$ . Consider the long exact sequence in cohomology induced by (22):

$$\cdots \to H^{q-1}(X,\mathfrak{Z}_{\ell}^{(p)}) \to H^{q-1}(X,S_{\ell-1}^{(p)}) \to H^{q}(X,S_{\ell}^{(p)}) \to H^{q}(X,\mathfrak{Z}_{\ell}^{(p)}) \to \cdots.$$

By Proposition 2.3, we have that for  $q \geq \ell$ ,  $H^q(X, \mathfrak{Z}^{(p)}_{\ell}) = 0$ , so that for  $q > \ell$  we have

$$H^{q-1}(X, S_{\ell-1}^{(p)}) \simeq H^q(X, S_{\ell}^{(p)}).$$

By varying  $\ell$ , we obtain a chain of isomorphisms

$$H^{q-p}(X, S_0^{(p)}) \xrightarrow{\sim} H^{q-p+1}(X, S_1^{(p)}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H^q(X, S_p^{(p)}),$$

which shows  $H^q(X, \bigwedge^p \mathcal{E}^{\vee}) \simeq \operatorname{Sym}^p W \otimes H^{q-p}(X, \mathcal{O}_X) = 0.$ 

Now, assume that q < p. By Serre duality  $H^q(X, \bigwedge^p \mathcal{E}^{\vee})$  is dual to  $H^{n-q}(X, \bigwedge^{n-p} \mathcal{E}^{\vee})$ , and n-q > n-p, so the latter vanishes by the above considerations.

Corollary 3.2. Let  $\mathcal E$  be a toric deformation of the tangent bundle. Then there is a canonical isomorphism

(23) 
$$H_{\mathcal{E}}^*(X) \simeq \bigoplus_k H^k(X, \bigwedge^k \mathcal{E}^{\vee}).$$

Remark 3.3. If we examine the particular case of  $k=1, \ell=1$ , we find that

(24) 
$$H^{1}(X, \mathcal{E}^{\vee}) \simeq H^{0}(X, \mathcal{O}_{X} \otimes W) \simeq W$$

by dualizing (17) to obtain

(25) 
$$0 \to \mathcal{E}^{\vee} \to Z \to W \otimes \mathcal{O} \to 0$$
 and using  $H^0(X, Z) = H^1(X, Z) = 0$ .

3.2. **Generators.** Now that we have a better idea of the linear structure of the algebra, we would like to identify a set of minimal generators. Corollary 3.2 and the fact that the cohomology of smooth projective toric varieties are generated by elements of  $H^1(X, \Omega_X^1)$  lead us to suspect that elements of  $H^1(X, \mathcal{E}^{\vee}) \simeq W$  generate  $H_{\mathcal{E}}^{\varepsilon}(X)$ .

As multiplication is defined using the cup product and the algebra is diagonal, it is in fact commutative. For  $1 \leq k \leq n$ , the cup product of k elements of  $H^1(X, \mathcal{E}^{\vee})$  is a linear map  $\operatorname{Sym}^k H^1(X, \mathcal{E}^{\vee}) \to H^k(X, \bigwedge^k \mathcal{E}^{\vee})$  that we will rewrite as

(26) 
$$m_k : \operatorname{Sym}^k W \to H^k(X, \bigwedge^k \mathcal{E}^{\vee}).$$

**Lemma 3.4.** The map  $m_k$  can be identified with the extension class in  $\operatorname{Ext}^k(\operatorname{Sym}^k W \otimes \mathcal{O}_X, \bigwedge^k \mathcal{E}^\vee) \simeq \operatorname{Hom}(\operatorname{Sym}^k W, H^k(X, \bigwedge^k \mathcal{E}^\vee))$  associated to the generalized Koszul complex (20).

*Proof.* We sketch the argument. First,  $m_1$  is identified with the extension class  $e \in \operatorname{Ext}^1(W \otimes \mathcal{O}, \mathcal{E}^{\vee})$  of (25), an exact sequence that can be reinterpreted as a quasi-isomorphism of  $\mathcal{E}^{\vee}$  with the complex

$$C^{\bullet}: 0 \to Z \to W \otimes \mathcal{O} \to 0$$

with Z in degree 0. Thus, e can be represented by the natural morphism of complexes  $f: W \otimes \mathcal{O} \to C^{\bullet}[1]$  whose only nontrivial component is the identity map on  $W \otimes \mathcal{O}$ . We can then tensor f with itself k times to get a morphism

$$f^{\otimes k}: W^{\otimes k} \otimes \mathcal{O} \to (C^{\bullet})^{\otimes k} [k].$$

However,  $(C^{\bullet})^{\otimes k}$  is quasiisomorphic to  $(\mathcal{E}^{\vee})^{\otimes k}$ , a statement that we can rewrite as an exact sequence

$$(27) 0 \to (\mathcal{E}^{\vee})^{\otimes k} \to (C^{\bullet})^{\otimes k} \to 0.$$

Noting that  $W^{\otimes k} \otimes \mathcal{O}$  is the degree k component of  $(C^{\bullet})^{\otimes k}$ , the above discussion identifies the extension class of (27) with

$$(m_1)^{\otimes k}: W^{\otimes k} \to H^k(X, \bigwedge^k \mathcal{E}^{\vee}).$$

Finally, there is a natural action of the symmetric group  $S_k$  on the complex (27) obtained by permuting the factors and inserting signs to preserve the structure as a complex. Then (20) is simply the  $S_k$ -invariant subcomplex of (27), and the extension class is  $m_k$ , the symmetrized version of  $m_1^k$ , as claimed.

Each short exact sequence in (22) induces coboundary maps

(28) 
$$\delta_j : H^j(X, S_j^{(k)}) \longrightarrow H^{j+1}(X, S_{j+1}^{(k)}),$$

which may be composed to a linear map

(29) 
$$\delta_{k-1} \circ \delta_{k-2} \cdots \circ \delta_0 : H^0(X, \operatorname{Sym}^k W \otimes \mathcal{O}_X) \to H^k(X, \bigwedge^k \mathcal{E}^{\vee}).$$

By first identifying  $H^0(X, \operatorname{Sym}^k W \otimes \mathcal{O}_X)$  with  $\operatorname{Sym}^k W$  and then applying the above composition of maps, we obtain precisely the linear map in (26). That is,

$$(30) m_k = \delta_{k-1} \circ \delta_{k-2} \circ \cdots \circ \delta_0.$$

**Proposition 3.5.** For all  $1 \le k \le n$ ,  $m_k$  is surjective.

*Proof.* Fix such a k; for all  $\ell$ ,  $H^{\ell}(X, \mathfrak{Z}_{\ell}^{(k)}) = 0$  by Proposition 2.3. In the long exact sequence in cohomology induced by the exact sequence (22), we obtain the following subsequences for each  $0 < \ell \le k$ :

$$(31) \qquad \cdots \longrightarrow H^{\ell-1}(X,\mathfrak{Z}_{\ell}^{(k)}) \longrightarrow H^{\ell-1}(X,S_{\ell-1}^{(k)}) \longrightarrow H^{\ell}(X,S_{\ell}^{(k)}) \longrightarrow 0$$

Thus, both the coboundary maps (28) and their composition (29) are surjective.  $\Box$ 

Remark 3.6. In fact, most of the coboundary maps are isomorphisms;  $H^{\ell-1}(X,\mathfrak{Z}_{\ell}^{(k)})$  is non-vanishing iff there is a primitive collection of order  $\ell$  in  $\Sigma$ , by Proposition 2.3. This observation allows us to characterize the kernel of the multiplication map: to find the relations amongst the generators.

3.3. **Relations.** For each primitive collection K we will exhibit an explicit element  $Q_K \in \ker m_k \subset \operatorname{Sym}^k W$ , where k = |K| as before. We will see later that  $H_{\mathcal{E}}^*(X)$  is the quotient of  $\operatorname{Sym} W$  by the ideal generated by the  $Q_K$ , with K varying over all primitive collections. For each K, we set

$$Z_K = \bigoplus_{\rho \in K} \mathcal{O}(-D_\rho),$$

and for each  $\ell \leq k$  we set

$$\mathfrak{Z}_{\ell}^{K} = \bigwedge^{\ell} Z_{K} \otimes \operatorname{Sym}^{k-\ell} W.$$

Then the  $\mathfrak{Z}_{\ell}^{K}$  are the terms of a subcomplex of (20). Note that

$$\mathfrak{Z}_{k}^{K}\simeq\mathcal{O}\left(-D_{K}\right),$$

and so  $H^{k-1}(X,\mathfrak{Z}_k^K)$  is isomorphic to  $\mathbb C$  by Proposition 2.3.

The following Lemma gives a procedure for computing  $Q_K$ .

#### Lemma 3.7.

- i) Given  $z_k \in Z^{k-1}(\mathfrak{J}_k^K)$ , then one can simultaneously choose  $z_\ell \in C^{\ell-1}(\mathfrak{J}_\ell^K)$  so that  $\delta(z_\ell) = \alpha_{\ell+1}(z_{\ell+1})$  for each  $1 \leq \ell \leq k-1$ , where the  $\alpha_i$  are given by (19).
- ii) Given choices of the  $z_{\ell}$  as above, then  $\alpha_1(z_1) \in Z^0(\operatorname{Sym}^k W \otimes \mathcal{O}) \simeq \operatorname{Sym}^k W$  depends only on the cohomology class  $[z_k] \in H^{k-1}(\mathfrak{Z}_k^K)$  of  $z_k$  and not on the choice of representative  $z_k$  of that cohomology class or any of the choices made for  $z_{\ell}$  above.
- iii)  $\alpha_1(z_1) \in \ker(m_k).$

*Proof.* For i), it is enough to show that the  $z_{\ell}$  can be successively chosen in descending order to satisfy the required property. If  $z_{\ell}$  has been chosen with  $\ell > 1$ , then

$$\delta(\alpha_{\ell}(z_{\ell})) = \alpha_{\ell}(\delta(z_{\ell})) = \alpha_{\ell}(\alpha_{\ell+1}(z_{\ell})) = 0,$$

so that  $\alpha_{\ell}(z_{\ell}) \in Z^{\ell-1}(\mathfrak{Z}_{\ell-1}^K)$  is a cocycle. Since  $H^{\ell-1}(\mathfrak{Z}_{\ell-1}^K) = 0$  by Proposition 2.3, it follows that  $\alpha_{\ell}(z_{\ell})$  is a coboundary so can be written as  $\delta(z_{\ell-1})$  for some  $z_{\ell-1} \in C^{\ell-2}(\mathfrak{Z}_{\ell-1}^K)$ .

For the first part of ii), note that  $\alpha_1(z_1) \in Z^0(\operatorname{Sym}^k W \otimes \mathcal{O})$  is clear by  $\delta(\alpha_1(z_1)) = \alpha_1(\delta(z_1)) = \alpha_1(\alpha_2(z_2)) = 0$ .

Next, we prepare for an induction on  $\ell$  by claiming that if  $z_{\ell+1}, \ldots, z_k$  are fixed and choices are only allowed to be made in  $z_1, \ldots, z_\ell$ , then  $\alpha_1(z_1)$  is independent of the allowed choices. The assertion of ii) is simply this claim for  $\ell = k$ .

Fix  $\ell < k$  and suppose that  $z_r$  has been chosen for all  $r > \ell$  and we want to choose  $z_\ell$  so that  $\delta(z_\ell) = \alpha_\ell(z_{\ell+1})$ . Once any  $z_\ell$  is chosen, then any other choice differs from  $z_\ell$  by a cocycle in  $Z^{\ell-1}(\mathfrak{Z}_\ell^K)$ , which is necessarily a coboundary  $\delta y_\ell$  for some  $(\ell-2)$ -cochain  $y_\ell \in C^{\ell-2}(\mathfrak{Z}_\ell^K)$ , since  $H^{\ell-1}(\mathfrak{Z}_\ell^K) = 0$ . If  $\ell = k$ , a separate argument is required, but the conclusion is the same: the only other choice for  $z_k$  is to modify it by the addition of a coboundary  $\delta y_k$ .

We start the induction with  $\ell=1$ . Since  $Z^0(\mathfrak{Z}_1^K)=H^0(\mathfrak{Z}_1^K)=0$ , there are no nontrivial cocyles, so that  $z_1$  is unique and the independence is trivially true for  $\ell=1$ . Now suppose that the claim is true for some  $\ell< k$  and we show that it is true for  $\ell+1$ . As noted above, we can only change the choice of  $z_{\ell+1}$  to  $z_{\ell+1}+\delta(y_{\ell+1})$  and then we see what that does to the rest of the computation. Then  $\alpha_{\ell+1}(z_{\ell+1})$  is replaced by

$$\alpha_{\ell+1}(z_{\ell+1} + \delta(y_{\ell+1})) = \alpha_{\ell+1}(z_{\ell+1}) + \delta(\alpha_{\ell+1}(y_{\ell+1})) = \delta(z_{\ell}) + \delta(\alpha_{\ell+1}(y_{\ell+1})).$$

Thus, we can continue the computation by replacing  $z_{\ell}$  by  $z_{\ell} + \alpha_{\ell+1}(y_{\ell+1})$ . Other choices for modifying  $z_{\ell}$  are possible, but by the inductive hypothesis, other choices won't affect  $\alpha_1(z_1)$ . At the next step,  $\alpha_{\ell}(z_{\ell})$  gets replaced by

$$\alpha_{\ell}(z_{\ell} + \alpha_{\ell+1}(y_{\ell+1})) = \alpha_{\ell}(z_{\ell}) + \alpha_{\ell}(\alpha_{\ell+1}(y_{\ell+1})) = \alpha_{\ell}(z_{\ell}),$$

and no change in  $z_{\ell-1}$  is required. The inductive hypothesis takes care of the rest.

For iii), let  $i_k: H^{k-1}(\mathfrak{Z}_k^{(k)}) \to H^{k-1}(S_{k-1}^{(k)})$  be the map in (31). Then we have constructed  $\alpha_1(z_1)$  so that

$$(\delta_{k-2} \circ \cdots \circ \delta_0) (\alpha(z_1)) = i_k ([z_k]).$$

By the exactness of (31), the image of  $i_k$  is the kernel of  $\delta_{k-1}$ . So

$$m_k\left(\alpha_1(z_1)\right) = \left(\delta_{k-1} \circ \cdots \circ \delta_0\right) \left(\alpha(z_1)\right) = 0$$

as claimed.  $\Box$ 

Note that Lemma 3.7 gives rise to a well-defined map

(32) 
$$\ell_K: H^{k-1}(\mathcal{O}(-D_K)) \to \operatorname{Sym}^k W, \qquad \ell_k([z_k]) = \alpha_1(z_1).$$

We now examine the form of the image of  $\ell_K$ . Let  $z \in H^{k-1}(\mathfrak{Z}_k^K)$ , and write  $K = \{\rho_1, \ldots, \rho_k\}$ .

Claim:  $\ell_K(z)$  has the form

(33) 
$$\sum_{m_1+m_2+\cdots+m_k=0} a_{\rho_1 m_1} \cdots a_{\rho_k m_k} \otimes p_{m_1 m_2 \cdots m_k},$$

where the  $p_{m_1\cdots m_k}$  lie in Sym<sup>k</sup>W and as before each  $m_i \in \Delta_{D_{\rho_i}} \cap M$ .

To see this, we follow the definition of  $\ell_K$ , choosing the  $z_\ell$  in decreasing order. After we have found  $\alpha_\ell(z_\ell)$  and then make a choice for  $z_{\ell-1}$ , the cocycle  $\alpha_{\ell-1}(z_{\ell-1})$  depends linearly on the coordinates  $\{a_{\rho m}\}$  of the moduli space of deformations  $\mathcal{E}$ . Since we compute  $\alpha_\ell(z_\ell)$  for  $\ell=1,\ldots,k$ , we conclude that  $\ell_K(z)$  is homogeneous of degree k as a function of the  $\{a_{\rho m}\}$ , of the form (33) without the qualifier  $m_1+\ldots+m_k=0$  in the sum. The restriction to  $m_1+\ldots+m_k=0$  occurs because  $a_{\rho_1 m_1}\cdots a_{\rho_k m_k}$  only arises in combination with a factor of  $\chi^{m_1+\ldots+m_k}$ , and  $\ell_K(z) \in H^0(\operatorname{Sym}^k W \otimes \mathcal{O})$ , which has pure weight 0. This proves the claim.

**Lemma 3.8.** Any primitive collection containing an edge  $\rho$  necessarily contains the edge  $\rho'$  if  $D_{\rho'}$  is linearly equivalent to  $D_{\rho}$ .

*Proof.* To see this, suppose  $K = \{\rho_1, \ldots, \rho_k\}$  is a primitive collection. Following [Bat93], we have that  $v_{\rho_1} + \ldots + v_{\rho_k}$  lies in the relative interior of a unique cone  $\sigma \in \Sigma$ , whose set of edges are disjoint from K. Letting the edges of  $\sigma$  be generated by primitive vectors  $w_1, \ldots, w_l$  we then have an identity

(34) 
$$v_{\rho_1} + \ldots + v_{\rho_k} = \sum_{i=1}^{l} c_i w_i$$

with all  $c_i > 0$ .

<sup>&</sup>lt;sup>2</sup>In making this assertion, we are using the fact the coboundary maps are independent of bundle moduli, so that  $z_{\ell}$  can recursively be chosen to have polynomial dependence on the  $\{a_{\rho m}\}$ , of degree  $k-\ell$ . It may only be possible to make choices of  $z_{\ell}$  on open subsets of bundle moduli; but since  $\ell_K$  is well-defined we conclude that there is polynomial dependence globally in bundle moduli.

Suppose there is an edge  $\rho'$  such that  $D_{\rho'}$  is linearly equivalent to one of the  $D_{\rho_i}$ . Then there exists an  $m \in M$  such that

$$\langle m, v_{\rho'} \rangle = 1, \quad \langle m, v_{\rho_i} \rangle = -1, \quad \langle m, v' \rangle = 0$$

where v' is any edge generator distinct from  $v_{\rho_i}$  or  $v_{\rho'}$ . Applying  $\langle m, \cdot \rangle$  to (34) we obtain

(35) 
$$\sum_{j=1}^{k} \langle m, v_{\rho_j} \rangle = \sum_{i=1}^{l} c_i \langle m, w_i \rangle.$$

The only negative term in this equation is  $\langle m, v_{\rho_i} \rangle$  on the left hand side. Therefore the left hand side must also contain a strictly positive term. This can only happen if v is one of the  $v_{\rho_j}$ .

For later use, we note that the same argument shows that the  $\{w_i\}$  are closed under the relation of linear equivalence of the corresponding divisors, and in fact linearly equivalent terms have to arise with identical coefficients  $c_i$  in (34) as the only way to get the identity  $0 = c_i - c_j$ .

We let [K] denote the set of equivalence classes of edges in K induced by linear equivalence of the corresponding divisors. Similarly, we let  $[K^-]$  denote the set of equivalence classes of the edges spanned by the  $w_i$  induced by linear equivalence of the corresponding divisors.

We now discuss the linear terms of E in more detail. For each  $\rho$ , write

$$E_{\rho} = \sum_{m \in \Delta_{D_{\sigma}} \cap M} a_{\rho m} \ x_{\rho} \prod_{\rho'} x_{\rho'}^{\langle m, v_{\rho}' \rangle}.$$

The linear monomials occurring in  $E_{\rho}$  come in two forms:  $x_{\rho}$ , and  $x_{\rho'}$  for certain  $\rho' \neq \rho$ . In the first case, the monomial  $x_{\rho}$  corresponds to  $m = 0 \in \Delta_{D_{\rho}}$ . In the second case,  $x_{\rho'}$  for  $\rho' \neq \rho$  can be a term in  $E_{\rho}$  if for some  $m \in M$  we have

$$\langle m, v_{\rho'} \rangle = 1$$
,  $\langle m, v_{\rho} \rangle = -1$ , and  $\langle m, v_{\rho''} \rangle = 0$ 

for all  $\rho'' \neq \rho, \rho'$ .

Note the linear monomials  $x_{\rho'}$  correspond to linear equivalences  $D_{\rho'} \sim D_{\rho}$ . The associated m is the one corresponding to the character whose divisor satisfies

$$(\chi^m) = D_{\rho'} - D_{\rho}.$$

Now partition the divisors  $D_{\rho}$  into linear equivalence classes, inducing a corresponding equivalence relation on  $\Sigma(1)$ : the linear part of  $E_{\rho}$  becomes

(36) 
$$E_{\rho}^{\text{lin}} = \sum_{\rho' \in [\rho]} a_{\rho\rho'} x^{\rho'},$$

where  $[\rho]$  is the equivalence class of  $\rho$  under the equivalence relation just described.

For each such equivalence class  $c = [\rho]$ , we define a  $|c| \times |c|$  matrix  $A_c$  with entries in W, where |c| is the size of the equivalence class, i.e. the number of divisors  $D_{\rho'}$  linearly equivalent to  $D_{\rho}$  (including  $D_{\rho}$  itself). The rows and columns of  $A_c$  can

be naturally identified with the edges comprising the equivalence class. The  $(\rho, \rho')$  entry of  $A_c$  is the coefficient  $a_{\rho\rho'}$  in (36) above.

**Definition 3.9.** The matrix  $A_c$  is the linear coefficient matrix associated to the divisor class corresponding to c.

Remark 3.10. Note that this matrix is precisely the one denoted  $A^a_{(\alpha)}$  in equation 2.5 of [MM08].

Remark 3.11. For  $\mathcal{E} = TX$  with its standard Euler sequence,  $A_c$  is diagonal with diagonal terms  $[D_{\rho}]$ .

**Lemma 3.12.** The image of  $\ell_K$  only depends on the linear terms of E.

*Proof.* We show that each term of (33) depends only on the linear terms. Since each  $m_j$  is in  $\Delta_{D_{\rho_j}}$ , we know that  $\langle m_j, v_{\rho_j} \rangle \geq -1$  while  $\langle m_j, v_{\rho} \rangle \geq 0$  for  $\rho \neq \rho_j$ . Then the vanishing of  $m_1 + \ldots + m_k$  implies that  $\sum_{i=1}^k \langle m_i, v_{\rho_\ell} \rangle = 0$  for all  $\ell$ , so we are left with two possibilities for each j:

- (1)  $m_i = 0$
- (2)  $m_i \neq 0$ , and there exists a unique i with  $i \neq j$  such that

$$\langle m_j, v_{\rho_\ell} \rangle = \begin{cases} 1 & \ell = i \\ -1 & \ell = j \\ 0 & \text{otherwise.} \end{cases}$$

As we have seen above, these  $m_i$  each are associated with linear terms.

We can at last explicitly describe the image of  $\ell_K$ .

**Lemma 3.13.** Let K be a primitive collection and  $[K] = \{[D_{\rho}] | \rho \in K\}$  as before. Up to  $\mathbb{C}$ -multiple, the image of  $\ell_K$  is

$$Q_K := \prod_{c \in [K]} \det A_c,$$

where the product is over the linear equivalence classes contained in K.

*Proof.* Our application of Theorem 2.1 to compute

$$H^{k-1}(X, \mathcal{O}(-D_K)) \simeq \widetilde{H}^{k-2}(S^{k-2}) \simeq \mathbb{C}$$

gives us more than the computation of the dimension of the cohomology group. If we choose an orientation of  $S^{k-2}$  (determined by an ordering of the edges in K), the fundamental class of  $S^{k-2}$  gives an almost canonical generator  $\gamma \in H^{k-1}(X, \mathcal{O}(-D_K))$  depending only on this orientation.

If we now choose any two edges  $\rho$  and  $\rho'$  in the same equivalence class and interchange them in the ordering of K, this changes the orientation of  $S^{k-2}$ , and hence the sign of  $\gamma$ , while switching  $E_{\rho}$  and  $E_{\rho'}$  and leaving everything else about the

input data unchanged. Then  $\ell_K(\gamma)$  is changed to  $-\ell_K(\gamma)$ . Thus, given an ordered primitive collection,  $\ell_K(\gamma) \in \operatorname{Sym}^k W$  is a degree k polynomial function of the  $a_{\rho m}$ , completely antisymmetric within each equivalence class c of edges corresponding to linear equivalence of divisors. We conclude that up to multiple it is necessarily the product of determinants associated with blocks of linearly equivalent  $D_i$ .

Since the expression  $\det A_c$  will arise frequently in our computations, we write

$$Q_c := \det A_c$$
.

For a perfect analogy with (11), we would have to express the polymology as a quotient of the homogeneous coordinate ring. Instead, we content ourselves with a description of the polymology as a quotient of  $\operatorname{Sym}^*W$ . We define the polymological analogue of the Stanley-Reisner ideal in  $\operatorname{Sym}^*W$ 

**Definition 3.14.** The Stanley-Reisner ideal of  $\mathcal{E}$  is

(37) 
$$SR(X, \mathcal{E}) = (Q_K \mid K \text{ a primitive collection of } \Sigma) \subset Sym^*W.$$

Remark 3.15. If  $\mathcal{E} = TX$ , then  $Q_K = \prod_{\rho \in K} [D_\rho]$ , and  $SR(X, T_X)$  is the image of the usual Stanley-Reisner ideal SR(X) in Sym<sup>\*</sup>W under the quotient (10).

We now have the main theorem of this section.

**Theorem 3.16.** Let X be a smooth projective toric variety,  $W = Pic(X) \otimes_{\mathbb{Z}} \mathbb{C}$ , and  $\mathcal{E} \to X$  a toric deformation of the tangent bundle of X. Then the polymology of  $\mathcal{E}$  is isomorphic as a graded algebra to

(38) 
$$H_{\mathcal{E}}^*(X) \simeq Sym^*W/SR(X,\mathcal{E})$$

*Proof.* The map  $m_k : \operatorname{Sym}^* W \to H_{\mathcal{E}}^*(X)$  is surjective by Proposition 3.5. We only have to note that its kernel is  $SR(X, \mathcal{E})$ , by (31), Remark 3.6, and Lemma 3.13.  $\square$ 

# 4. Quantum sheaf cohomology

In the following, we denote the Mori cone of X by  $NE(X) \subset H_2(X,\mathbb{R})$ , and its integral points by  $NE(X)_{\mathbb{Z}} = NE(X) \cap H_2(X,\mathbb{Z})$ . Since for a smooth toric variety the Mori cone is generated by the curves associated to the cones in  $\Sigma(n-1)$  (see e.g. [CLS11]), NE(X) is polyhedral.

4.1. **Moduli space.** For each  $\beta \in NE(X)_{\mathbb{Z}}$ , we describe the GLSM moduli space variety  $X_{\beta}$  after [Wit93, MP95]. We will think of  $X_{\beta}$  as a compactification of the space of holomorphic maps  $\mathbb{P}^1 \to X$  of class  $\beta$ , although in the GLSM the  $\beta$  arise instead as a natural index for the topological type of the gauge bundle on  $\mathbb{P}^1$ .

Fix a  $\beta \in NE(X)_{\mathbb{Z}}$ , and let  $d_{\rho}^{\beta} = D_{\rho} \cdot \beta$ . An actual map  $f : \mathbb{P}^1 \to X$  can be described in homogeneous coordinates as

(39) 
$$x_{\rho} = f_{\rho}, \qquad f_{\rho} \in H^{0}(\mathbb{P}^{1}, \mathcal{O}(d_{\rho}^{\beta})).^{3}$$

Accordingly, put

$$\mathbb{C}_{\beta} = \bigoplus_{\rho} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_{\rho}^{\beta})).$$

There is a natural action of  $G = \operatorname{Hom}(\operatorname{Pic}(X), \mathbb{C}^*)$  on  $\mathbb{C}_{\beta}$ , where G acts on  $f_{\rho}$  as multiplication by  $g([D_{\rho}])$ . Thinking of a point in  $\mathbb{C}_{\beta}$  as a map  $\mathbb{C}^2 \to \mathbb{C}^{\Sigma(1)}$ , we define a set  $Z_{\beta} \subset \mathbb{C}_{\beta}$  to be those tuples of sections defining a map whose image is contained in  $Z(\Sigma) \subset \mathbb{C}^{\Sigma(1)}$ . One can easily check that

$$(40) X_{\beta} = (\mathbb{C}_{\beta} - Z_{\beta}) / G$$

is a toric variety.<sup>4</sup>

We can alternatively describe  $X_{\beta}$  as the toric variety  $X_{\Sigma_{\beta}}$  associated to a fan  $\Sigma_{\beta}$ . Let  $(t_0, t_1)$  be homogeneous coordinates on  $\mathbb{P}^1$  and for each  $\rho$  with  $d_{\rho}^{\beta} \geq 0$  write

(41) 
$$f_{\rho} = \sum_{i=0}^{d_{\rho}^{\beta}} f_{\rho_i} t_0^i t_1^{d_{\rho}^{\beta} - i}.$$

Let  $\mathbb{C}_{\beta}^* \subset \mathbb{C}_{\beta}$  be the subset on which all  $f_{\rho_i}$  are nonzero. Then  $T_{\beta} = \mathbb{C}_{\beta}^*/G \subset X_{\beta}$  is a dense torus acting on  $X_{\beta}$ , giving it the structure of a toric variety. Defining the lattice of 1-parameter subgroups of  $T_{\beta}$  as  $N_{\beta}$ , we have a fan  $\Sigma_{\beta}$  in  $N_{\beta} \otimes \mathbb{R}$  whose edges  $\Sigma_{\beta}(1)$  naturally correspond to  $T_{\beta}$ -invariant divisors  $D_{\rho_i}$  defined by  $f_{\rho_i} = 0$ .

We would like to identify the edges in  $\Sigma_{\beta}(1)$  as  $\rho_i$  with  $\rho \in \Sigma(1)$  and  $i = 0, \ldots, d_{\rho}^{\beta}$ . Then the  $f_{\rho_i}$  would be naturally identified with the homogeneous coordinates of  $X_{\beta}$ .

There is however a subtlety in that not all of the divisors  $D_{\rho_i}$  defined by  $f_{\rho_i} = 0$  correspond to edges in the fan  $\Sigma_{\beta}$ , since it can happen that  $D_{\rho_i}$  can be empty. Before explaining how these arise in general, an example will clarify the phenomenon.

We consider the Hirzebruch surface  $F_n$  described as a 2-dimensional toric variety whose edges  $\rho_1 \dots, \rho_4$  are respectively spanned by the vectors

$$v_1 = (1,0), v_2 = (-1,n), v_3 = (0,1), v_4 = (0,-1).$$

Related aspects of this example are also discussed in [DGKS11].

Let  $\beta = D_{\rho_3}$ , the class of the -n curve. Then the  $d_{\rho_i}^{\beta}$  are respectively 1, 1, -n, 0 for  $i = 1, \ldots, 4$ , and so

$$\mathbb{C}_{\beta} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^2 \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}).$$

<sup>&</sup>lt;sup>3</sup>In the GLSM, the  $f_{\rho}$  are identified with the zero modes of certain charged chiral fields in the theory.

<sup>&</sup>lt;sup>4</sup>In the GLSM, the constraint leading to the avoidance of  $Z_{\beta}$  arises from FI terms.

Note that  $f_{\rho_3} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-n))$  is identically zero, so that  $Z_{\beta}$  contains the hyperplane  $f_{(\rho_4)_0} = 0$  owing to the primitive collection  $K = \{\rho_3, \rho_4\}$  in the fan  $\Sigma$  of  $F_n$ . It follows that  $D_{(\rho_4)_0}$  is empty, hence  $(\rho_4)_0$  is not in the fan  $\Sigma_{\beta}$ . In fact,  $\Sigma_{\beta}$  is readily identified with the standard toric fan for  $\mathbb{P}^3$ , with the four edges  $(\rho_i)_j$  for i = 1, 2 and j = 0, 1. We have  $\Sigma_{\beta}(1) = \{(\rho_i)_j \mid i = 1, 2, \text{ and } j = 0, 1\}$ .

We now explain when  $D_{\rho_i}$  can be empty. If  $d_{\rho}^{\beta} < 0$ , then there are no  $f_{\rho_i}$ . If  $d_{\rho}^{\beta} > 0$ , then  $f_{\rho_i}$  can be zero while  $f_{\rho}$  from (41) is not identically zero, so  $D_{\rho_i}$  is certainly nonempty. It follows that  $D_{\rho_i}$  is empty precisely when  $d_{\rho}^{\beta} = 0$  and for some primitive collection K of  $\Sigma$  containing  $\rho$  we have  $d_{\rho'}^{\beta} < 0$  for all  $\rho' \neq \rho$  in K.

We will consider these  $\rho_0$  as degenerate edges of  $\Sigma_{\beta}$ .

To formalize these considerations and facilitate a uniform treatment, we introduce the notation

(42) 
$$\widehat{\Sigma}_{\beta}(1) = \left\{ \rho_i \mid \rho \in \Sigma(1), \ 0 \le i \le d_{\rho}^{\beta} \right\}.$$

In (42),  $\widehat{\Sigma}_{\beta}(1)$  is a set of formal symbols. We can and will identify those  $\rho_i$  whose associated divisor  $D_{\rho_i}$  is nonempty with edges  $\rho_i$  of the fan  $\Sigma_{\beta}(1)$ . No confusion should result from this slight abuse of notation. Thus,  $\widehat{\Sigma}_{\beta}(1)$  is an enhancement of the set of edges of the fan  $\Sigma_{\beta}$  by the degenerate edges.

In the case of  $F_n$ , we have  $\widehat{\Sigma}_{\beta}(1) = \Sigma_{\beta}(1) \cup \{(\rho_4)_0\}.$ 

With this understanding, we can now alternatively specify the fan  $\Sigma_{\beta}$  by specifying the primitive collections within the set  $\widehat{\Sigma}_{\beta}(1)$  of enhanced edges.

Recalling that  $Z(\Sigma)$  is the union of the linear subspaces  $L_K$ , we infer that  $Z_{\beta}$  is the union over all K of the subspaces of sections defining maps whose images are contained in  $L_K \subset \mathbb{C}^{\Sigma(1)}$ . If  $K = \{\rho_1, \ldots, \rho_k\}$ , then this subspace is defined by imposing  $f_{\rho_{j_i}} = 0$  for  $1 \leq j \leq k$ .

Accordingly, for each primitive collection  $K \subset \Sigma(1)$ , define  $K_{\beta} \subset \widehat{\Sigma}_{\beta}(1)$  as the set of all edges corresponding to those in K:

$$K_{\beta} = \left\{ \left\{ \rho_i \in \widehat{\Sigma}_{\beta}(1) \right\} \mid \rho \in K \right\}.$$

It is straightforward to see that the  $K_{\beta}$  are the primitive collections for  $X_{\beta}$ , with a natural extension of the notion of primitive collection to  $\widehat{\Sigma}_{\beta}(1)$ . This is essentially because we can recognize  $Z_{\beta}$  as the union of the linear subspaces  $L_{K_{\beta}}$ . Furthermore, the fan  $\Sigma_{\beta}$  consists of the cones spanned by all collections of edges in  $\Sigma_{\beta}(1)$  that do not contain any primitive collection  $K_{\beta}$ . These general constructions of toric geometry identify  $Z_{\beta}$  with  $Z(\Sigma_{\beta})$ . Note that if  $\rho_0$  is a degenerate edge, then by

this definition, the singleton set  $\{\rho_0\}$  is a primitive collection. But this is exactly what we want, since  $\rho_0$  is not part of the actual fan.

For later use, we introduce the following numerical function.

**Definition 4.1.** Define the function  $h^0: \mathbb{Z} \to \mathbb{Z}$  by

$$h^{0}(x) = \begin{cases} x+1 & x \ge -1 \\ 0 & x \le -1, \end{cases}$$

or more concisely,  $h^0(x) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(x)) = \max(0, x+1).$ 

Note that for each edge  $\rho \in \Sigma(1)$  there are  $h^0(d^{\beta}_{\rho})$  edges in  $\Sigma_{\beta}(1)$ .

**Proposition 4.2.** For all  $\beta \in NE(X)_{\mathbb{Z}}$ ,  $X_{\beta}$  is a smooth projective toric variety.

*Proof.* See [CFK10, Example 7.2.3] for a proof that  $X_{\beta}$  is projective, and we have already explained that it is toric. For smoothness, it is elementary to show that a simplicial toric variety X is smooth if and only if, upon writing  $X = (\mathbb{C}^{\Sigma(1)} - Z(\Sigma))/G$ , the action of G on  $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$  is free. Thus, the smoothness of X implies that G acts freely on  $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ .

This easily implies that the action of G on  $\mathbb{C}^{\Sigma_{\beta}(1)} - Z(\Sigma_{\beta})$  is also free as follows. Let  $f \in \mathbb{C}^{\Sigma_{\beta}(1)} - Z(\Sigma_{\beta})$  with f expressed in terms of the  $f_{\rho_i}$  as in (41). Then for any primitive collection K and edge  $\rho \in K$ , we can't have  $f_{\rho_i} = 0$  for all i by the definition of  $K_{\beta}$ . It follows that the  $f_{\rho_i}(t_0, t_1)$  can't all vanish for generic  $(t_0, t_1) \in \mathbb{C}^2$ , hence there exists  $(t_0, t_1) \in \mathbb{C}^2$  with  $f(t_0, t_1) \notin Z$ , where  $f(t_0, t_1) \in \mathbb{C}^{\Sigma(1)}$  has coordinates  $f_{\rho}(t_0, t_1)$ . If in addition such an f is a fixed point, then  $f(t_0, t_1) \in \mathbb{C}^{\Sigma(1)} - Z(\Sigma)$  would also be a fixed point, contradicting the smoothness of X. Therefore  $X_{\beta}$  is smooth as well.

Remark 4.3.  $X_0 = X$ ; the moduli space of constant maps is X itself.

4.2. **Induced sheaf.** Since  $X_{\beta}$  is a smooth projective toric variety, we have its Euler exact sequence

$$(43) 0 \to \mathcal{O}_{X_{\beta}} \otimes_{\mathbb{Z}} \operatorname{Pic}(X_{\beta})^{\vee} \xrightarrow{F_{0}} \bigoplus_{\rho_{i} \in \Sigma_{\beta}(1)} \mathcal{O}_{X_{\beta}}(D_{\rho_{i}}) \to T_{X_{\beta}} \to 0,$$

where the  $\rho_i^{\text{th}}$  component of the morphism  $F_0$  is  $f_{\rho_i} \otimes [D_{\rho_i}]$ .

Recalling that  $D_{\rho_0}$  is empty for a degenerate edge  $\rho_0$ , by adding trivial line bundles to each of the first two nonzero bundles in (43) for each degenerate edge, we obtain a modification of the Euler sequence of  $X_{\beta}$ 

$$(44) 0 \to \mathcal{O}_{X_{\beta}} \otimes W^{\vee} \xrightarrow{\hat{F}_{0}} \bigoplus_{\rho_{i} \in \widehat{\Sigma}_{\beta}(1)} \mathcal{O}_{X_{\beta}}(D_{\rho_{i}}) \to T_{X_{\beta}} \to 0,$$

where the  $\rho_i^{\text{th}}$  component of the morphism  $\hat{F}_0$  is  $f_{\rho_i} \otimes [D_{\rho}]$ .

We add a few words of clarification on the relationship between (43) and (44), even though it is not essential for the sequel.

Let

$$\widehat{\mathbb{C}}_{\beta} = \widehat{\bigoplus}_{\rho} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_{\rho}^{\beta})),$$

where the hat over the direct sum means that we are omitting the components of  $\mathbb{C}_{\beta}$  associated with degenerate edges. Let  $G' \subset G$  be the subgroup which acts as the identity on the linear subspace of  $\mathbb{C}_{\beta}$  defined by the vanishing of the  $f_{\rho_i}$  with  $\rho_i \in \Sigma_{\beta}(1)$  (rather than  $\widehat{\Sigma}_{\beta}(1)$ ). Then we have

(45) 
$$X_{\beta} = \left(\widehat{\mathbb{C}}_{\beta} - \left(Z_{\beta} \cap \widehat{\mathbb{C}}_{\beta}\right)\right) / G',$$

where  $\widehat{\mathbb{C}}_{\beta}$  is viewed as a subspace of  $\mathbb{C}_{\beta}$  in the natural way. In fact, (45) is just the usual description of  $X_{\beta}$  as a quotient constructed from the fan  $\Sigma_{\beta}(1)$ .

By (16),  $\operatorname{Pic}(X)$  is canonically isomorphic to  $\operatorname{Hom}(G,\mathbb{C}^*)$  and  $\operatorname{Pic}(X_\beta)$  is canonically isomorphic to  $\operatorname{Hom}(G',\mathbb{C}^*)$ . The inclusion  $G' \hookrightarrow G$  therefore induces a mapping  $\operatorname{Pic}(X) \to \operatorname{Pic}(X_\beta)$  which is needed in justifying the claimed relationship between (43) and (44). It is the modified version (44) of the toric euler sequence that gets deformed by a deformation  $\mathcal{E}$  of TX. With slight abuse of notation, we will refer to deformations of the map  $\hat{F}_0$  as giving rise to toric deformations of the tangent bundle of  $X_\beta$ . No confusion should result.

We now define the induced sheaf  $\mathcal{E}_{\beta}$  precisely as dictated by the GLSM. On X, we defined a bundle  $\mathcal{E}$  in terms of sections  $E_{\rho}$  of  $\mathcal{O}_X(D_{\rho}) \otimes W$ , for each  $\rho \in \Sigma(1)$ . We shall now associate to these sections corresponding sections of  $\mathcal{O}_{X_{\beta}}(D_{\rho_i}) \otimes W$ , leading to a toric deformation  $\mathcal{E}_{\beta}$  of the tangent bundle of  $X_{\beta}$  in the modified sense just explained.

We express each  $E_{\rho}$  in terms of homogeneous coordinates; for emphasis we write this as  $E_{\rho} = E_{\rho}(x)$ . Over  $X_{\beta}$  we then make the substitutions  $x_{\rho} = f_{\rho}(t_0, t_1)$  for each  $\rho \in \Sigma(1)$  to obtain expressions that we abbreviate as  $E_{\rho}(f(t))$ . We now collect powers of  $t_0$  and  $t_1$ , writing the result as

(46) 
$$E_{\rho}(f(t)) = \sum_{i=0}^{d_{\rho}^{\beta}} E_{\rho_i}(f) t_0^i t_1^{d_{\rho}^{\beta} - i}.$$

The W-valued expressions  $E_{\rho_i}(f)$  are then interpreted as the components of a toric deformation  $\mathcal{E}_{\beta}$  of the tangent bundle of  $X_{\beta}$  in the modified sense, defined by the exact sequence

$$(47) 0 \to \mathcal{O}_{X_{\beta}} \otimes W^{\vee} \xrightarrow{\hat{F}} \bigoplus_{\rho_{i} \in \widehat{\Sigma}_{\beta}(1)} \mathcal{O}_{X_{\beta}}(D_{\rho_{i}}) \to \mathcal{E}_{\beta} \to 0,$$

where the components of  $\hat{F}$  are as described above.

**Proposition 4.4.**  $\mathcal{E}_{\beta}$  is locally-free if  $\mathcal{E}$  is.

*Proof.* The local freeness of  $\mathcal{E}$  is equivalent to the assertion that the  $E_{\rho}(x)$  span W for all  $x \in \mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ —this is just the surjectivity of (18).

Now let  $f \in C_{\beta} - Z_{\beta}$ . Taking a generic  $t = (t_0, t_1) \in \mathbb{C}^2$  as in the proof of Proposition 4.2, it follows that  $f(t_0, t_1) \in \mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ , hence the  $E_{\rho}(f(t))$  span W. But (46) says that  $E_{\rho}(f(t))$  is in the span of the  $E_{\rho_i}(f)$ . It follows immediately that the  $E_{\rho_i}(f)$  span W as well, hence  $E_{\beta}$  is locally free.  $\square$ 

We now turn to the computation of the polymology of  $(X_{\beta}, \mathcal{E}_{\beta})$ . Since  $\mathcal{E}_{\beta}$  is a toric deformation of the tangent bundle of  $X_{\beta}$  in the modified sense, its polymology algebra can be computed by the same method as in Section 3, resulting in a description of its polymology as a quotient of the symmetric algebra of W. The only change that is needed is to consider degenerate edges. Recall that for a degenerate edge  $\rho_0$ ,  $D_{\rho_0}$  is empty. So we have to supplement Proposition 2.3 with

$$H^{\ell}(X_{\beta}, \mathcal{O}(-D_{\rho_0})) = \left\{ \begin{array}{cc} \mathbb{C} & \ell = 0 \\ 0 & \ell > 0 \end{array} \right.$$

for degenerate edges  $\rho_0$ . The reasoning in Section 3 produces an element  $Q_{[\rho_0]} \in W$  associated to the primitive collection  $\{\rho_0\}$  which is in the kernel of the map  $W \to H^1(X_\beta, \mathcal{E}_\beta^\vee)$  arising from the long exact sequence associated to the dual of (47). We let  $\widehat{SR}(X_\beta, \mathcal{E}_\beta)$  be the ideal in Sym\*W generated by  $SR(X_\beta, \mathcal{E}_\beta)$  and the  $Q_{[\rho_0]}$  just described.

To compute  $\widehat{SR}(X_{\beta}, \mathcal{E}_{\beta})$ , we just need to compute the  $Q_{K_{\beta}}$ .

Fix a  $\rho \in \Sigma(1)$  and write

$$E_{\rho} = \sum_{\rho}' A_{\rho\rho'} x_{\rho'} + \cdots,$$

where the sum is over edges  $\rho'$  with  $D_{\rho'}$  linearly equivalent to  $D_{\rho}$  and the  $\cdots$  represent the omitted nonlinear terms. Said differently, A is the linear coefficient matrix  $A_c$  associated to the linear equivalence class c containing  $\rho$ . Then

$$E_{\rho}(f) = \sum_{\rho'} A_{\rho\rho'} f_{\rho'} + \cdots = \sum_{\rho', i} A_{\rho\rho'} f_{\rho'} t_0^i t_1^{d_{\beta} - i} + \cdots,$$

so that

(48) 
$$E_{\rho_i} = \sum_{\rho'} A_{\rho\rho'} f_{\rho'_i} + \cdots.$$

Denoting the analogue of  $A_c$  for  $\mathcal{E}_{\beta}$  by  $(A_{\beta})_c$ , equation (48) says that  $(A_{\beta})_c$  has a block diagonal form consisting of  $h^0(d_{\beta})$  copies of  $A_c$ . Thus

$$\det (A_{\beta})_c = Q_c^{h^0(d_{\beta})}.$$

It follows immediately that

$$Q_{K_{\beta}} = \prod_{c \in [K]} Q_c^{h^0(D_c \cdot \beta)}$$

and

$$H_{\mathcal{E}_{\beta}}^* = \operatorname{Sym}^* W / \widehat{SR}(X_{\beta}, \mathcal{E}_{\beta}),$$

where

$$\widehat{SR}(X_{\beta}, \mathcal{E}_{\beta}) = (Q_{K_{\beta}} \mid K \text{ a primitive collection of } \Sigma) \subset \operatorname{Sym}^* W.$$

Note that if K is a primitive collection of  $\Sigma$  containing an edge  $\rho$  of  $\Sigma$  with  $\rho_0$  a degenerate edge of  $\Sigma_{\beta}$ , then from (49) we get  $Q_{K_{\beta}} = Q_{[\rho]}$ , where  $[\rho] = \{\rho\}$  is the linear equivalence class of  $\rho$ . But this is precisely the linear part of  $\widehat{SR}(X_{\beta}, \mathcal{E}_{\beta})$  by our earlier discussion.

As with the classical case, we will define the correlation functions in the sector labelled by  $\beta$  as elements of the one-dimensional vector space  $H^{n_{\beta}}(X_{\beta}, \bigwedge^{n_{\beta}} \mathcal{E}_{\beta})$ , where  $n_{\beta}$  is the dimension of  $X_{\beta}$ . The precise definition will be spelled out later, but first we have to grapple with the normalization issue.

The correlation functions will be obtained as usual by adding the contributions over the sectors  $\beta$ . However, since these contributions live in *different* one-dimensional vector spaces, we will describe a distingushed space  $H^*$ , together with a collection of isomorphisms

$$H_{\mathcal{E}_{\beta}}^{n_{\beta}} \simeq H^*$$

for each  $\beta$ , in which the contributions will be summed. This will be accomplished by assembling the  $H_{\mathcal{E}_{\beta}}^{n_{\beta}}$  into a direct system in the next section.

4.3. **Direct system of polymologies.** For every  $\beta \in H_2(X,\mathbb{Z})$ , we have constructed an induced deformation  $\mathcal{E}_{\beta}$  of the toric Euler sequence on  $X_{\beta}$ . The algebra  $H_{\mathcal{E}_{\beta}}^*(X_{\beta})$  is generated by elements of  $H^1(X_{\beta}, \mathcal{E}_{\beta}^{\vee}) \simeq W$ .

We now construct a direct system from these polymologies and show that the one-dimensional spaces  $H^{n_{\beta}}(X_{\beta}, \bigwedge^{n_{\beta}} \mathcal{E}_{\beta}^{\vee})$  are preserved by the maps of the direct system, hence by restriction also give a direct system. We will show that the direct limit is a one-dimensional vector space, in which the correlation functions take their values.

For each c corresponding to a linear equivalence class of divisors  $D_{\rho}$ , we put  $d_c^{\beta} = d_{\rho}^{\beta}$ , for any  $\rho$  in the equivalence class.

**Definition 4.5.** For classes  $\beta, \beta' \in H_2(X, \mathbb{Z})$ , we say that  $\beta'$  dominates  $\beta$  if  $\beta' - \beta$  is effective and  $h^0(d_c^{\beta'}) \geq h^0(d_c^{\beta})$  for all linear equivalence classes c of the irreducible toric divisors  $D_{\rho}$ .

If  $\beta'$  dominates  $\beta$ , we define the expression

(50) 
$$R_{\beta'\beta} = \prod_{c} Q_c^{h^0(d_c^{\beta'}) - h^0(d_c^{\beta})} \in \text{Sym}^* W.$$

**Lemma 4.6.** Suppose that  $\beta'$  dominates  $\beta$ . Then

$$R_{\beta'\beta}\left(SR(X_{\beta},\mathcal{E}_{\beta})\right) \subset SR(X_{\beta'},\mathcal{E}_{\beta'}).$$

*Proof.* We will show that for each primitive collection K,  $R_{\beta'\beta}Q_{K_{\beta}}$  is a multiple of  $Q_{K_{\beta'}}$ . This will suffice to prove the lemma, by the definitions of  $SR(X_{\beta}, \mathcal{E}_{\beta})$  and  $SR(X_{\beta'}, \mathcal{E}_{\beta'})$ .

For this, it suffices to compare the powers of  $Q_c$  occurring in  $R_{\beta'\beta}Q_{K_{\beta}}$  and  $Q_{K_{\beta'}}$ , for each c. If  $c \in [K]$ , then the exponent of  $Q_c$  in  $R_{\beta'\beta}Q_{K_{\beta}}$  is

$$h^{0}(d_{c}^{\beta'}) - h^{0}(d_{c}^{\beta}) + h^{0}(d_{c}^{\beta}) = h^{0}(d_{c}^{\beta'}),$$

which is the exponent of  $Q_c$  in  $Q_{K_{a'}}$ .

If  $c \notin [K]$ , then the exponent of  $Q_c$  in  $R_{\beta'\beta}Q_{K_\beta}$  is  $h^0(d_c^{\beta'}) - h^0(d_c^{\beta})$ , the exponent of  $Q_c$  in  $Q_{K_{\beta'}}$  is 0, and the required inequality holds by the dominance assumption.  $\square$ 

Whenever  $\beta'$  dominates  $\beta$ , multiplication by  $R_{\beta'\beta}$  induces a well-defined map

$$f_{\beta'\beta}: H^*_{\mathcal{E}_\beta}(X_\beta) \to H^*_{\mathcal{E}_{\beta'}}(X_{\beta'})$$

by Lemma 4.6. It is straightforward to verify that the  $\{f_{\beta'\beta}\}$  form a direct system. We have to show that the maps are compatible and that any  $\beta_1$  and  $\beta_2$  are dominated by some  $\beta$ . Compatibility is obvious, and given any  $\beta_1$  and  $\beta_2$ , choose a  $\beta_3$  effective that has positive intersection with each  $D_c^5$  and set  $\beta = \beta_1 + \beta_2 + n\beta_3$  for some  $n \gg 0$ . Then  $\beta$  dominates both  $\beta_1$  and  $\beta_2$ .

For simplicity of notation, let  $H_{\mathcal{E}_{\beta}}^{n_{\beta}}$  be the degree  $n_{\beta}$  part of  $\operatorname{Sym}^*W/(SR(X_{\beta}), \mathcal{E}_{\beta})$  (which is canonically isomorphic to  $H^{n_{\beta}}(X_{\beta}, \bigwedge^{n_{\beta}} \mathcal{E}_{\beta}^{\vee})$ ).

# Lemma 4.7.

i) If  $\beta'$  dominates  $\beta$ , then  $f_{\beta'\beta}(H_{\mathcal{E}_{\beta}}^{n_{\beta}}) \subset H_{\mathcal{E}_{\beta'}}^{n_{\beta'}}$ . Thus the maps

$$g_{\beta'\beta} := f_{\beta'\beta}|_{H^{n_\beta}_{\mathcal{E}_\beta}} : H^{n_\beta} \to H^{n_{\beta'}}_{\mathcal{E}_{\beta'}}$$

also form a direct system.

ii) If  $X_{\beta}$  and  $X_{\beta'}$  are nonempty, then  $g_{\beta'\beta}$  is an isomorphism for deformations  $\mathcal{E}$  sufficiently close to TX in the moduli space of toric deformations of TX.

It follows immediately from Lemma 4.7 that the direct limit

$$H^* := \lim_{\stackrel{\longrightarrow}{g}} H^{n_{\beta}}_{\mathcal{E}_{\beta}}$$

is a 1-dimensional vector space, and the induced maps  $i_{\beta}: H_{\beta}^* \to H^*$  are isomorphisms. The correlation functions will all take values in  $H^*$ .

<sup>&</sup>lt;sup>5</sup>An intersection of ample divisors suffices.

Proof of Lemma 4.7. The emptiness of  $X_{\beta}$  is equivalent to  $D_c \cdot \beta < 0$  for all c that are part of some fixed primitive collection K, by the definition of the primitive collections for  $X_{\beta}$ .

For i), we just have to show that the cohomology degrees are compatible with  $g_{\beta'\beta}$ .

Noting that  $Q_c$  has degree |c|, the number of divisors  $D_{\rho}$  in the corresponding linear equivalence class, we see that

$$\deg R_{\beta'\beta} = \sum_{c} |c| \left( h^0(d_c^{\beta'}) - h^0(d_c^{\beta}) \right)$$
$$= \sum_{\rho} \left( h^0(d_{\rho}^{\beta'}) - h^0(d_{\rho}^{\beta}) \right).$$

Thus we must show

(51) 
$$n_{\beta'} = n_{\beta} + \sum_{\rho} \left( h^0(d_{\rho}^{\beta'}) - h^0(d_{\rho}^{\beta}) \right).$$

Before computing  $n_{\beta} = \dim X_{\beta}$  we note that for a general toric variety X we have

$$\dim X = |\Sigma(1)| - h^2(X) = \left(\sum_{a} 1\right) - h^2(X),$$

as follows from the quotient description (14) and dim  $G = \operatorname{rank}(Pic(X)) = h^2(X)$ . Applying the same calculation to  $X_{\beta}$ , we count the edges in  $\Sigma_{\beta}(1)$  and recall that  $h^2(X_{\beta}) = h^2(X)$  to conclude

(52) 
$$n_{\beta} = \dim X_{\beta} = \left(\sum_{\rho} h^{0}(d_{\rho}^{\beta})\right) - h^{2}(X),$$

and (51) follows immediately.

For ii), we just have to show that the map  $g_{\beta'\beta}$  of one-dimensional vector spaces is nonzero.

Claim: If  $R_{\beta'\beta} \notin SR(X_{\beta'}, \mathcal{E}_{\beta'})$ , then  $g_{\beta'\beta}$  is an isomorphism.

To justify the claim, the hypotheses can be restated as saying that  $[R_{\beta'\beta}]$  is nonzero in  $H_{\mathcal{E}_{\beta'}}^*$ . Since  $X_{\beta'}$  is smooth and  $\mathcal{E}_{\beta'}$  is locally free,  $H_{\mathcal{E}_{\beta'}}^*$  satisfies Poincaré duality. Thus, there exists an element  $p \in \operatorname{Sym}^* W$  such that  $[pR_{\beta'\beta}]$  is a nonzero element of  $H_{\mathcal{E}_{\beta'}}^{n_{\beta'}}$ . Thus  $g_{\beta'\beta}(p) \neq 0$ , justifying the claim.

It therefore suffices to show that  $R_{\beta'\beta} \notin SR(X_{\beta'}, \mathcal{E}_{\beta'})$ . We do this by first verifying it for  $\mathcal{E} = TX$ . Once we show that, we have proven the first part of the lemma in a neighborhood of TX by the closedness of the ideal membership condition.

We now assume that  $\mathcal{E} = T_X$  and identify the polymology of  $\mathcal{E}$  with the cohomology of X. We will show that  $g_{\beta'\beta}$  applied to the cohomology class of a point of  $X_{\beta}$  is

the cohomology class of a point of  $X_{\beta'}$ .<sup>6</sup> Since the class of a point is nontrivial, it follows that  $g_{\beta'\beta}$  is nonzero. In this case, it is easy to see that  $Q_c = [\prod_{\rho \in c} x_\rho]$  by looking at the components of the toric Euler sequence (8).

Recall that  $S = \mathbb{C}[x_{\rho} \mid \rho \in \Sigma(1)]$  is the homogeneous coordinate ring of X. We have the ring homorphism  $S \to \operatorname{Sym}^* W$  defined by taking  $x_{\rho}$  to the cohomology class of  $D_{\rho}$ . Starting with a polynomial p, we can take its image in  $\operatorname{Sym}^* W$  and then take a further quotient by  $\operatorname{SR}(X)$  to get a cohomology class  $[p] \in H^*(X)$ . We also let  $S_{\beta}$  be the homogeneous coordinate ring of  $X_{\beta}$ , with a similar map to  $\operatorname{Sym}^* W$  and we use the notation  $[p]_{\beta}$  for the image of a polynomial in  $H^*(X_{\beta})$ .

If  $\beta'$  dominates  $\beta$ , then  $S_{\beta}$  can be regarded as a subring of  $S_{\beta'}$  in a natural way as follows. Recall that  $S_{\beta}$  is generated by the variables  $x_{\rho_1}, \ldots, x_{\rho_{d_{\rho}^{\beta'}}}$  while  $S_{\beta'}$  is generated by the variables  $x_{\rho_1}, \ldots, x_{\rho_{d_{\rho}^{\beta'}}}$ . The dominance assumption implies that  $d_{\rho}^{\beta'} \geq d_{\rho}^{\beta}$  for each  $\rho$ , so the shared nomenclature of the variables provides a natural embedding of  $S_{\beta}$  in  $S_{\beta'}$ . Furthermore  $R_{\beta'\beta}$  is simply the image in SymW of the product

$$m = \prod_{\rho} \prod_{i=d_{\rho}^{\beta}+1}^{d_{\rho}^{\beta'}} x_{\rho_i}$$

corresponding to the additional edges added in going from  $X_{\beta}$  to  $X_{\beta'}$ .

When we multiply  $R_{\beta'\beta}$  as represented by m by the class of a point represented by a product of variables in  $S_{\beta}$  corresponding to a maximal cone  $\sigma$  of  $X_{\beta}$ , it is clear that multiplication by m corresponds to simply appending the additional edges  $\rho_{d_{\rho}^{\beta}+1}, \ldots, \rho_{d_{\rho}^{\beta'}}$  needed to complete  $\sigma$  to a maximal cone of  $X_{\beta'}$ . Since the resulting monomial represents the class of a point, a nonzero cohomology class, the resulting monomial in  $S_{\beta'}$  cannot be in the Stanley-Reisner ideal.

Part of the above argument appeared in [MP95].

4.4. Correlation Functions. In this section, we will define the correlation functions. In the half-twisted GLSM, correlation functions in sector  $\beta$  can be nonzero only if the operators have degree  $c_1(X) \cdot \beta + \dim(X)$ . Since  $c_1(X) = \sum_{\rho} D_{\rho}$ , it follows that

$$c_1(X) \cdot \beta = \sum_{\rho} d_{\rho}^{\beta},$$

hence

(53) 
$$c_1(X) \cdot \beta + \dim(X) = \sum_{\rho} \left( d_{\rho}^{\beta} + 1 \right) - h^2(X),$$

The formula (53) plays the role of the virtual or expected dimension of Gromov-Witten theory.

<sup>&</sup>lt;sup>6</sup>This matching of point classes appeared in the context of ordinary cohomology in [MP95], and was part of our motivation for the definition of  $R_{\beta'\beta}$ .

Note that (52) and (53) differ only in that  $h^0(d^{\beta}_{\rho})$  in (52) is replaced by  $d^{\beta}_{\rho} + 1$  in (53). These are in fact equal, unless  $d^{\beta}_{\rho} \leq -2$  for some  $\rho$ . Such a situation is the analogue of excess dimension in Gromov-Witten theory. In our situation, we have both the excess dimension of  $X_{\beta}$  and the excess rank of  $\mathcal{E}_{\beta}$ .

In this case, to compensate, we need something to play the role of the obstruction classes of Gromov-Witten theory. These were called four-fermi terms in [KS06] because of how they arose in the path integral.

Define  $h^1(x) = h^1(\mathcal{O}_{\mathbb{P}^1}(x))$  in analogy with Definition 4.1. Then our formula for the four-fermi terms is

(54) 
$$F_{\beta} = \prod_{c} Q_c^{h^1(d_c^{\beta})},$$

where the product is over all linear equivalence classes c of the divisors  $D_{\rho}$ .

At last, we can define the correlation functions in sector  $\beta$ . Let

$$p \in \operatorname{Sym}^{c_1(X) \cdot \beta + \dim X} W$$
.

A simple computation of degree shows that  $pF_{\beta} \in H_{\mathcal{E}_{\beta}}^{n_{\beta}}$ : the degree of  $pF_{\beta}$  is

$$\deg(pF_{\beta}) = c_1(X) \cdot \beta + \dim X + \sum_c |c| h^1(d_c^{\beta})$$

$$= \sum_{\rho} (d_{\rho}^{\beta} + 1) - h^2(X) + \sum_{\rho} h^1(d_{\rho}^{\beta})$$

$$= \sum_{\rho} h^0(d_{\rho}^{\beta}) - h^2(X)$$

$$= n_{\beta}$$

In the third line of the above computation, we have used the identity

(55) 
$$h^{0}(d_{\rho}^{\beta}) - h^{1}(d_{\rho}^{\beta}) = d_{\rho}^{\beta} + 1,$$

which is Riemann Roch for  $\mathbb{P}^1$ .

Finally, the correlation function is defined as

$$\langle p \rangle_{\beta} = i_{\beta} (p F_{\beta}) \in H^*,$$

where  $i_{\beta}$  was defined immediately following the statement of Lemma 4.7. Following our discussion at the beginning of this section, if  $p \in \operatorname{Sym}^{d}W$  with  $d \neq c_{1}(X) \cdot \beta + \dim X$ , we define  $\langle p \rangle_{\beta} = 0$ . By design, all correlation functions live in the same one-dimensional vector space  $H^{*}$ , so can be added over  $\beta$ . To formalize the sum, we recall one version of the Novikov ring.

**Definition 4.8.** The Novikov ring  $\mathbb{C}[q^{\beta}]$  of X is the ring generated over  $\mathbb{C}$  by the formal expressions  $q^{\beta}$  for each  $\beta \in NE(X)$ , subject to the relations  $q^{\beta}q^{\beta'}=q^{\beta+\beta'}$ .

For any  $p \in \operatorname{Sym}^* W$  we define the correlation function

$$\langle p \rangle = \sum_{\beta} \langle p \rangle_{\beta} q^{\beta} \in \operatorname{Sym}^* W \otimes \mathbb{C}[q^{\beta}].$$

4.5. Quantum Cohomology Ring. In the quasi-topological sector of the half-twisted GLSM, as in all quantum field theories where correlation functions are independent of the insertion point, an operator  $\mathcal{O}$  is the trivial operator iff

$$\langle \mathcal{O}, \mathcal{O}_1, \dots, \mathcal{O}_k \rangle = 0$$

for all operators  $\mathcal{O}_1, \ldots, \mathcal{O}_k$ . Accordingly, the trivial operators with coefficients in the Novikov ring form an ideal in the ring  $\operatorname{Sym}^*W \otimes \mathbb{C}[q^{\beta}]$ , which we suggestively call the *quantum Stanley-Reisner ideal* and write as  $QSR(X, \mathcal{E})$ .

**Definition 4.9.** The quantum sheaf cohomology of  $(X, \mathcal{E})$  is the ring

$$QH_{\mathcal{E}}^*(X) = \left(Sym^*W \otimes \mathbb{C}[q^{\beta}]\right)/QSR(X,\mathcal{E}).$$

By definition, if we set all  $q^{\beta}=0$ , the correlation functions become the classical correlation functions described in Section 3 and the quantum Stanley-Reisner ideal  $QSR(X,\mathcal{E})$  becomes the ordinary Stanley-Reisner ideal  $SR(X,\mathcal{E})$ . Thus,  $QSR(X,\mathcal{E})$  is a deformation of  $SR(X,\mathcal{E})$ . Accordingly, we expect  $QSR(X,\mathcal{E})$  to be generated by deformations of the generators  $Q_K$  of  $SR(X,\mathcal{E})$ . In fact, passing to the localization to make the comparison, the relations in [MM09] are of just this form. We will view these as predictions for the generators in our set-up and then prove that they are correct.

Fixing a primitive collection K, we rewrite (34) as

$$\sum a_{\rho} v_{\rho} = 0,$$

where  $a_{\rho} = 1$  for each  $\rho \in K$ . Then there exists a unique  $\beta_K \in H_2(X, \mathbb{Z})$  such that

$$d_{\rho}^{\beta_K} = D_{\rho} \cdot \beta_K = a_{\rho} \qquad \forall \rho.$$

Furthermore, the  $\beta_K$  generate the cone of effective curves. Details of these assertions can be found in [CLS11].

Then there is proposed a relation

(58) 
$$\prod_{c \in [K]} Q_c = q^{\beta_K} \prod_{c \in [K^-]} Q_c^{-d_c^{\beta_K}}$$

Since the left hand side of (58) is just  $Q_K$ , (58) says that the quantities

$$Q_K - q^{\beta_K} \prod_{c \in [K^-]} Q_c^{-d_c^{\beta_K}}$$

are in  $QSR(X,\mathcal{E})$  and specialize to the generator  $Q_K$  of  $SR(X,\mathcal{E})$  when the  $q^{\beta}$  are set to 0.

By definition, (58) is equivalent to the identity of correlation functions

(59) 
$$\langle Y \prod_{c \in [K]} Q_c \rangle_{\beta + \beta_K} = \langle Y \prod_{c \in [K^-]} Q_c^{-d_c^{\beta_K}} \rangle_{\beta}$$

for any  $Y \in \text{Sym}^*W$  and  $\beta \in H_2(X, \mathbb{Z})$ .

**Theorem 4.10.** The quantum cohomology relations (58) hold for all primitive collections K.

*Proof.* We show (59) for any  $Y \in \text{Sym}^*W$ . Choosing  $\beta'$  dominating both  $\beta$  and  $\beta + \beta_K$ , we have to show the equality

(60) 
$$R_{\beta',\beta+\beta_K} F_{\beta+\beta_K} Y \prod_{c \in K} Q_c = R_{\beta'\beta} F_{\beta} Y \prod_{c \in [K^-]} Q_c^{-D_c \cdot \beta_K}$$

as elements of the quotient ring

$$\operatorname{Sym}^* W/SR(X_{\beta'}, \mathcal{E}_{\beta'}).$$

In fact, we will see that (60) holds in  $Sym^*W$ . For this it suffices to show

(61) 
$$R_{\beta',\beta+\beta_K} F_{\beta+\beta_K} \prod_{c \in K} Q_c = R_{\beta'\beta} F_{\beta} \prod_{c \in [K^-]} Q_c^{-D_c \cdot \beta_K}$$

Both sides of (61) expand to products of powers of the  $Q_c$ , so we just have to check the exponents of each  $Q_c$ . We break this up into three cases, according to whether  $d_c^{\beta_K}$  is positive, negative, or zero, or equivalently,  $c \in [K]$ ,  $c \in [K^-]$ , or c in neither [K] nor  $[K^-]$ . In any case, we note that

$$(62) d_c^{\beta+\beta_K} = d_c^{\beta} + d_c^{\beta_K}$$

If  $d_c^{\beta_K} = 0$ , then the required equality of exponents is

$$h^0(d_c^{\beta'}) - h^0(d_c^{\beta + \beta_K}) + h^1(d_c^{\beta + \beta_K}) = h^0(d_c^{\beta'}) - h^0(d_c^{\beta}) + h^1(d_c^{\beta}).$$

However, in this case,  $d_c^{\beta} = d_c^{\beta + \beta_K}$  by (62) and equality is clear.

If  $d_c^{\beta_K} > 0$  then  $c \in [K]$  and the required equality of exponents is

$$h^{0}(d_{c}^{\beta'}) - h^{0}(d_{c}^{\beta+\beta_{K}}) + h^{1}(d_{c}^{\beta+\beta_{K}}) + 1 = h^{0}(d_{c}^{\beta'}) - h^{0}(d_{c}^{\beta}) + h^{1}(d_{c}^{\beta}).$$

The equality follows immediately from (62),  $d_c^{\beta} = 1$ , and two applications of (55) (one time with  $\beta$  replaced by  $\beta + \beta_K$ ).

Finally, if  $d_c^{\beta_K} < 0$ , then  $c \in [K^-]$  and we have to show

$$h^{0}(d_{c}^{\beta'}) - h^{0}(d_{c}^{\beta+\beta_{K}}) + h^{1}(d_{c}^{\beta+\beta_{K}}) = h^{0}(d_{c}^{\beta'}) - h^{0}(d_{c}^{\beta}) + h^{1}(d_{c}^{\beta}) - d_{c}^{\beta_{K}},$$

which is easily verified in the same way.

From physics, we expect this result to generalize to complete intersections in toric varieties. Suppose that  $Y \subset X$  is a complete intersection in a toric variety X. Then the tangent bundle of Y is the cohomology of a monad. The cohomology of a small deformation of this monad will be a vector bundle  $\mathcal{E}$  on Y. Then  $\mathrm{Sym}^*W$  generates a subalgebra of the polymology of  $\mathcal{E}$ , which we call the *toric polymology* of  $\mathcal{E}$ . We write

(63) 
$$H_{\mathcal{E}}^*(X)^{\text{toric}} = (\operatorname{Sym}^* W) / \operatorname{SR}(X, \mathcal{E}),$$

where for present purposes  $SR(X, \mathcal{E})$  is defined by (63).

Conjecture 4.11. There is a toric quantum sheaf cohomology ring  $QH_{\mathcal{E}}^*(X)^{\text{toric}}$  which is of the form

$$QH_{\mathcal{E}}^*(X)^{\text{toric}} = (\operatorname{Sym}^* W) / \operatorname{QSR}(X, \mathcal{E}),$$

where  $QSR(X, \mathcal{E})$  specializes to  $SR(X, \mathcal{E})$  after setting all the  $q^{\beta}$  to zero.

Note that we have proven this conjecture for Y = X.

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